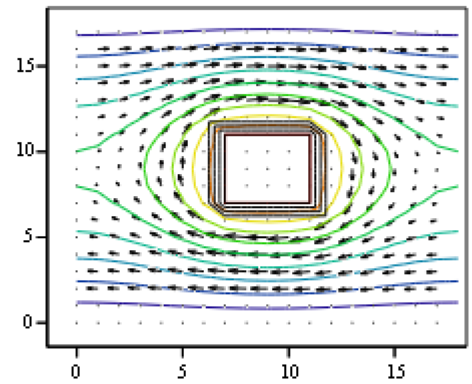
Fluid flow II: The stream function



$(Ux2, Uy2), Psi2, Initial2$

Flow around an object

This exercise is a continuation of the Fluid flow I exercise. Read that exercise for an introduction.

It is possible to obtain solutions for incompressible flow using the stream function ψ rather than the potential function Φ. The relation between the fluid velocity **u** and the stream function ψ is:

$$u_x = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

Note that flow left-to-right ($u_x > 0$) requires ψ being larger at the top (largest y value). ψ at interior points will be found using the relaxation method so that ψ satisfies Poisson's equation. Finding ψ this way gives us a simple solution for the flow. Many other flow solutions are possible, thus our answer is not unique.

Try it: Show that the divergence of the fluid velocity is zero indicating that ψ describes an incompressible fluid.

The grid

There is no reason to assign values to the x_i and y_j that are on the grid. It is sufficient to know the number of x grid points imax, the number of y grid points jmax, and the grid spacings Δx and Δy.

$imax := 18$ The number of x grid points. These are on the horizontal axis.

$jmax := 18$ The number of y grid points. These are on the vertical axis.

The subscripts will have the values: $i := 0 .. imax$ $j := 0 .. jmax$

$\Delta x := 1$ $\Delta y := 1$ The grid spacing is assumed unity in each direction.

Boundary condition Boolean matrix:

We will use a boundary condition matrix BB that has "1" at the nozzle wall and outside the nozzle walls. Inside the boundary, BB = 0. Fluid is in the cells having BB = 0 at any corner. The program loop gives the nozzle a slope of 1/3 so that the opening at the small end of the nozzle is 1/3 of the opening at the large end. The subscript i runs from the bottom of the nozzle to the top (see the figures) and the subscript j runs from left to right.

The bondary conditon matrix:

$$
BB := \left|
\begin{array}{l}
BB_{imax,jmax} \leftarrow 0 \\
\text{for } i \in 0..imax \\
\quad \text{for } j \in 0..jmax \\
\quad\quad \left|
\begin{array}{l}
BB_{i,j} \leftarrow 1 \\
BB_{i,j} \leftarrow 0 \quad \text{if } \dfrac{j}{jmax} \le \dfrac{i}{3 \cdot imax} \vee \dfrac{jmax - j}{jmax} \le \dfrac{i}{3 \cdot imax}
\end{array}
\right. \\
BB
\end{array}
\right.
$$

Boundary condition matrix for the top half of the nozzle:

$BB^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...

The initial gradient in ψ:

At the top of the nozzle ψ will have a positive value and at the bottom of the nozzle ψ will have a negative value, so that there is a gradient in ψ that causes flow to the right. The gradient will be made approximately equal to 1 in our dimensionless units system which means that ψ = 9 at the top of the nozzle and y = -9 at the bottom (when jmax = 18). The space outside the nozzle will be given the same value as the nearest nozzle boundary.

$$
Initial := \left|
\begin{array}{l}
Initial_{imax,jmax} \leftarrow 0 \\
\text{for } i \in 0..imax \\
\quad \text{for } j \in 0..jmax \\
\quad\quad \left|
\begin{array}{l}
Initial_{i,j} \leftarrow -0.5 \cdot jmax \quad \text{if } j > \dfrac{jmax}{2} \wedge BB_{i,j} = 0 \\
Initial_{i,j} \leftarrow 0.5 \cdot jmax \quad \text{if } j < \dfrac{jmax}{2} \wedge BB_{i,j} = 0
\end{array}
\right. \\
Initial
\end{array}
\right.
$$

Initial condition matrix for the top half of the nozzle:

$Initial^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
1	0	0	0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
2	0	0	0	0	0	0	9	9	9	9	9	9	9	9	9	9	9	9	9
3	0	0	0	0	0	0	0	0	0	9	9	9	9	9	9	9	9	9	9
4	0	0	0	0	0	0	0	0	0	0	0	0	9	9	9	9	9	9	9
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	9	9	9
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...

The program loop

The program loop will create the Psi function by averaging every point in the interior with with four surroundings points. The number of iterations is set equal to several times the number of points.

$iters := 3 \cdot imax \cdot jmax$ The number of iterations.

```
Psi(BB, Initial) := | Psi ← Initial
                    | Avg ← Initial
                    | for k ∈ 0 .. iters
                    |   | i ← 0
                    |   | for j ∈ 1 .. jmax − 1
                    |   |   | Avg[i,j] ← 0.25·(2·Psi[i+1,j] + Psi[i,j−1] + Psi[i,j+1])  if BB[i,j] ≠ 0
                    |   |   | Avg[i,j] ← Avg[i,j]  otherwise
                    |   | for i ∈ 1 .. imax − 1
                    |   |   for j ∈ 1 .. jmax − 1
                    |   |     | Avg[i,j] ← 0.25·(Psi[i−1,j] + Psi[i+1,j] + Psi[i,j−1] + Psi[i,j+1])  if BB[i,j] ≠ 0
                    |   |     | Avg[i,j] ← Avg[i,j]  otherwise
                    |   | i ← imax
                    |   | for j ∈ 1 .. jmax − 1
                    |   |   | Avg[i,j] ← 0.25·(2·Psi[i−1,j] + Psi[i,j−1] + Psi[i,j+1])  if BB[i,j] ≠ 0
                    |   |   | Avg[i,j] ← Avg[i,j]  otherwise
                    |   | Psi ← Avg
                    | Psi
```

Avg is defined as a simple average of the Psi values at surrounding points. If we had not assumed $\Delta x = \Delta y = 1$, then the average have to be a properly weighted average.

At the left and right boundaries, i = 0 and i = imax, symmetric boundary conditions are applied. It is assumed that the point just outside the boundary has the same y value and the point just inside the boundary. These boundary conditions are applied by having two extra loops in the program for the boundary points i = 0 and i = imax.

The answer matrix Psi: $\mathrm{Psi1} := \mathrm{Psi}(\mathrm{BB}, \mathrm{Initial})$

$\mathrm{Psi1}^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
1	8.3	8.3	8.5	9	9	9	9	9	9	9	9	9	9	9	9	9
2	7.4	7.5	7.6	7.9	8.1	8.3	9	9	9	9	9	9	9	9	9	9
3	6.5	6.6	6.7	6.8	7	7.3	7.6	7.9	8.2	9	9	9	9	9	9	9
4	5.5	5.6	5.6	5.8	5.9	6.2	6.4	6.7	7	7.4	7.7	8.1	9	9	9	9
5	4.5	4.5	4.6	4.7	4.8	5	5.2	5.4	5.6	5.9	6.2	6.6	7.1	7.4	7.9	9
6	3.4	3.4	3.5	3.5	3.6	3.8	3.9	4.1	4.3	4.5	4.7	5	5.3	5.6	6	6.6
7	2.3	2.3	2.3	2.4	2.4	2.5	2.6	2.7	2.9	3	3.2	3.3	3.6	3.8	4.1	4.4
8	1.1	1.2	1.2	1.2	1.2	1.3	1.3	1.4	1.4	1.5	1.6	1.7	1.8	1.9	2	2.2
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...

The velocity field

The vector field for the velocities is found from ψ using the definitions at the beginning of the exercise.

$ii := 1 .. imax - 1$ $\quad jj := 1 .. jmax - 1$ Define a new set of indices that does not include the boundary points.

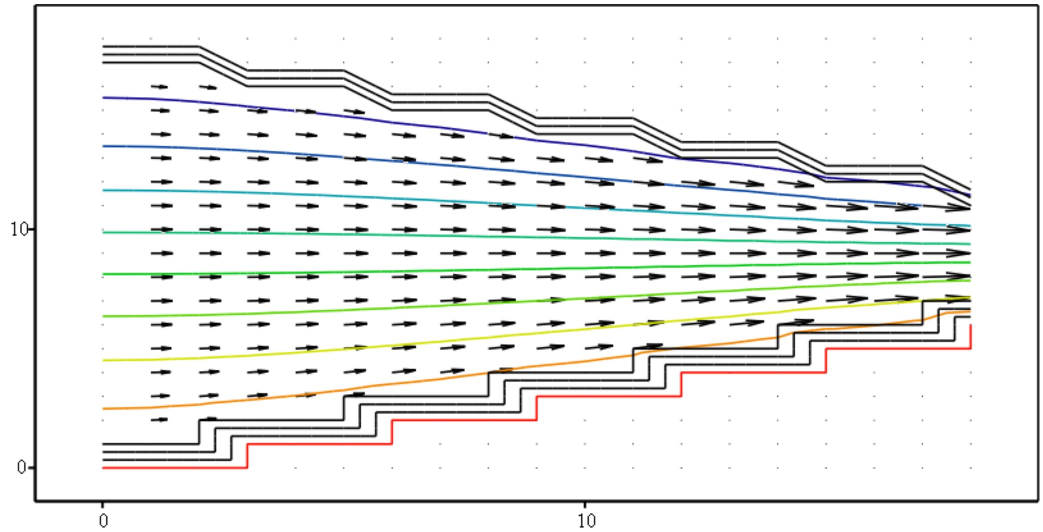
$Ux_{imax, jmax} := 0$ $\quad Uy_{imax, jmax} := 0$ Initialize the flow vectors

The finite-differencing above is centered on the point ii,jj. If this point is in the fluid, then the sum of the neighboring BB values is equal to 4. If this sum is not 4, the velocity vector is set to zero so that it does not appear in the plot.

$$Ux_{ii,jj} := \mathrm{if}\left(BB_{ii,jj-1} + BB_{ii,jj+1} + BB_{ii-1,jj} + BB_{ii+1,jj} = 4, -\frac{Psi1_{ii,jj+1} - Psi1_{ii,jj-1}}{2 \cdot \Delta y}, 0\right)$$

$$Uy_{ii,jj} := \mathrm{if}\left[BB_{ii,jj-1} + BB_{ii,jj+1} + BB_{ii-1,jj} + BB_{ii+1,jj} = 4, \frac{\left(Psi1_{ii+1,jj}\right) - \left(Psi1_{ii-1,jj}\right)}{2 \cdot \Delta x}, 0\right]$$

Vector field plot of **u** and contour plots of Psi1 and the boundary matrix BB



(Ux, Uy), Psi1, BB

The stream function been plotted as a contour plot using a small number of contour lines.

Is there conservation of fluid?

To find the amount of fluid per unit time entering the left boundary, we integrate u_x from the bottom to the top:

$$\int_1^2 u_x dy = \int_1^2 \frac{d\psi}{dy} dy = \psi_2 - \psi_1$$

where 1 and 2 refer to the lower boundary and upper boundary, respectively.

This integral has the same value for all vertical paths through the nozzle because ψ has the same value at all points on the upper nozzle boundary and the same (lower) value at all points in the lower nozzle boundary. The length of the flow vectors increases from left to right as the area of the nozzle decreases.

Circular flow about an object

A second example of flow is clockwise flow around an object. For this case, we will use a duct of constant cross section (no nozzle) and place a circular object within it. Recall that our equations are two-dimensional, hence the circular object is a cylinder (infinitely long) oriented perpendicular to the plane of the plots. The radius of the object will be made 1/6 of the height of the duct.

The boundary points are the top and bottom of the domain (the duct) and the boundary of the object. The points on the boundary of the duct and the boundary of the circular object have their BB value set to 0.

$$
\begin{aligned}
BB2 := \; & BB2_{imax,jmax} \leftarrow 0 \\
& \text{for } i \in 0..imax \\
& \quad \text{for } j \in 0..jmax \\
& \qquad BB2_{i,j} \leftarrow 1 \\
& \qquad BB2_{i,j} \leftarrow 0 \;\; \text{if } j = 0 \vee j = jmax \\
& \qquad BB2_{i,j} \leftarrow 0 \;\; \text{if } \left[\left(\frac{imax}{2} - i\right)^2 + \left(\frac{jmax}{2} - j\right)^2\right] < \left(\frac{jmax}{6}\right)^2 \\
& BB2
\end{aligned}
$$

The intial values of ψ are made zero at the duct walls and -1 at the circular object. The equal values at the top and bottom boundaries specify that there is not net flow in the duct from left to right. A value of -1 on the object creates a positive gradient near the upper duct wall and a negative gradient near the lower duct wall. Hence the flow is clockwise. At the left and right boundaries, there is flow to the right in the upper half plane and flow to the left in the lower half plane.

$$
\begin{aligned}
Initial2 := \; & Initial_{imax,jmax} \leftarrow 0 \\
& \text{for } i \in 0..imax \\
& \quad \text{for } j \in 0..jmax \\
& \qquad Initial_{i,j} \leftarrow 0 \\
& \qquad Initial_{i,j} \leftarrow 1 \;\; \text{if } \left[\left(\frac{imax}{2} - i\right)^2 + \left(\frac{jmax}{2} - j\right)^2\right] < \left(\frac{jmax}{6}\right)^2 \\
& Initial
\end{aligned}
$$

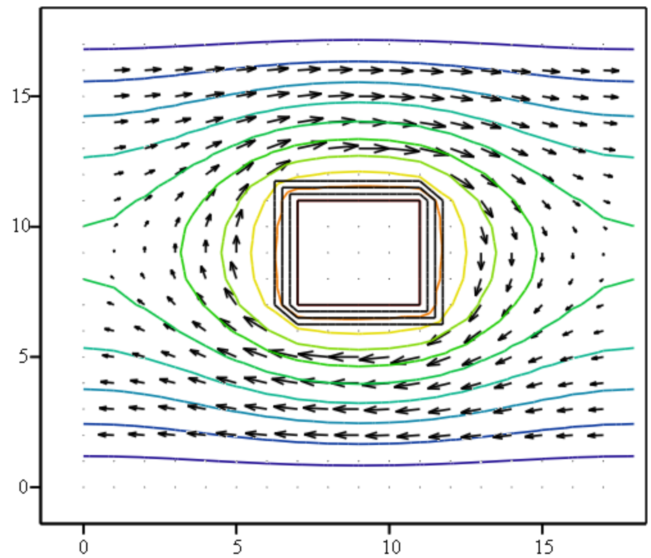
Find the ψ values using the Psi function for the new conditions $Psi2 := Psi(BB2, Initial2)$

Find the new velocity field from the new ψ

$$Ux2_{ii,jj} := \text{if}\left(BB2_{ii,jj-1} + BB2_{ii,jj+1} + BB2_{ii-1,jj} + BB2_{ii+1,jj} = 4, -\frac{Psi2_{ii,jj+1} - Psi2_{ii,jj-1}}{2\cdot\Delta y}, 0\right)$$

$$Uy2_{ii,jj} := \text{if}\left(BB2_{ii,jj-1} + BB2_{ii,jj+1} + BB2_{ii-1,jj} + BB2_{ii+1,jj} = 4, \frac{Psi2_{ii+1,jj} - Psi2_{ii-1,jj}}{2\cdot\Delta x}, 0\right)$$

Plot of flow in a duct with a cylindrical object



(Ux2, Uy2), Psi2, Initial2

The circular object appears as a polygon in a contour plot because of the small number of grid points.

Try it: Change the ψ values at the duct walls so that there is a net flow from left to right superimposed upon the circular flow.

Try it: Rewrite the exercise so that the flow is correct if Δx and Δy are arbitrary.

Reference

A. M. Kuethe and J. D. Schetzer, *Foundation of Aerodynamics* (Wiley, New York, 1967), chapter 2.