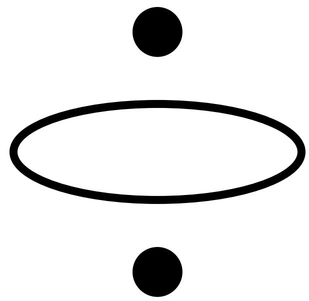
Paul trap, ponderomotive force, and Mathieu's equation

The Paul trap is an experimental device for storing electrons or ions. There is a circular electrode, often a ring, with spherical electrodes above and below the center. The top and bottom electrodes are connected together and an AC potential is applied between the pair and the ring. The potential is approximately given by



$$\phi(r,z) := \left(\frac{z^2}{2} - \frac{r^2}{4}\right)\cdot\cos(\Omega\cdot t)$$

where Ω is the frequency: $\Omega := 1$ $t := 0$

The potential satisfies Laplace's equation in cylindrical coordinates.
The components of the electric field are

$$E_z(z,t) := -z\cdot\cos(\Omega\cdot t) \qquad E_r(r,t) := \frac{r}{2}\cdot\cos(\Omega\cdot t)$$

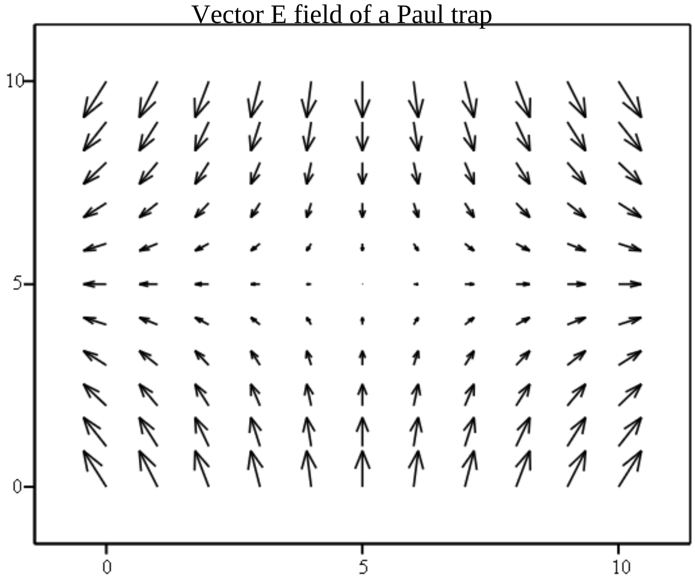
To plot the vector E field we need to put the vector components into matrices EZ and ER

$$i := 0..10 \qquad \rho_i := \frac{i-5}{5} \qquad j := 0..10 \qquad \zeta_j := \frac{j-5}{5}$$

$$EZ_{i,j} := E_z\left(\zeta_j, t\right) \qquad ER_{i,j} := E_r\left(\rho_i, t\right)$$

For plotting,
ρ is the radial coordinate
ζ is the axial coordinate

Vector E field of a Paul trap



(ER, EZ)

$\rho_i =$

-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

$\zeta_j =$

-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Equations of motion

We will put a particle in the trap with a charge and mass given by:

$q := .45 \quad m := 1$

The phase space coordinates of the particle will be in a 6- vector Z:

$ORIGIN := 1$

$$Z := \begin{pmatrix} .2 \\ .1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} r \\ z \\ dr/dt \\ dz/dt \end{matrix}$$

We start the particle at r,z = 0.2, 0.1.

The derivatives for the Runge Kutta routine are:

$$DZ(t,Z) := \begin{pmatrix} Z_3 \\ Z_4 \\ \frac{q}{m} \cdot E_r(Z_1,t) \\ \frac{q}{m} \cdot E_z(Z_2,t) \end{pmatrix}$$

Our time interval should be divided more finely than the period of the AC voltage.

Let t be the total time interval $t := 300$

The number of iterations we will use is about 4 per radian period or 8 π per oscillation.

$npoints := ceil(4 \cdot |\Omega| \cdot t)$ *ceil* is a rounding function to make integers

$npoints = 1200$ This is calculated, not guessed. It is always a good idea to base npoints on something other than a wild guess. A not-wild guess is a time interval dt of 1/4 of the characteristic time of the problem, in this case $1/\Omega$. The number of these intervals in the time t is equal to 4 Ω t.

Now integrate:

$F := rkfixed(Z, 0, t, npoints, DZ)$

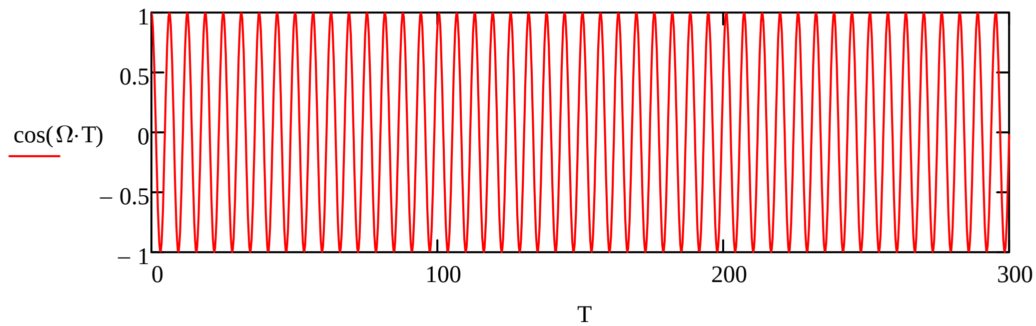
t r z dr/dt dz/dt

F =

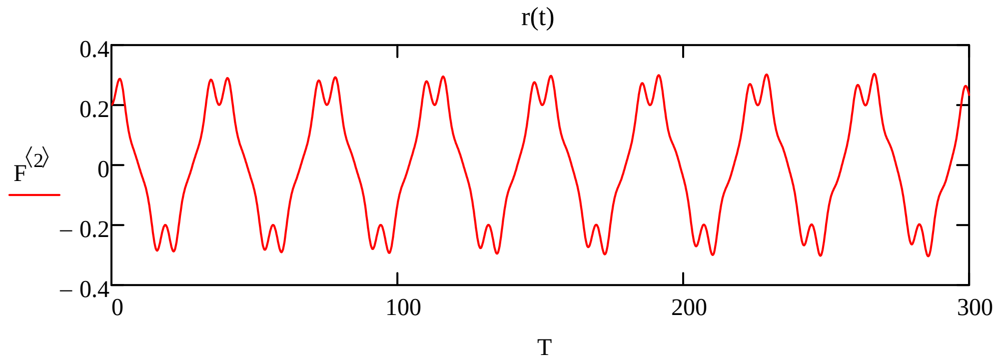
	1	2	3	4	5
1	0	0.2	0.1	0	0
2	0.25	0.201	0.099	0.011	-0.011
3	0.5	0.206	0.095	0.022	-0.021
4	0.75	0.212	0.088	0.031	-0.03
5	1	0.221	0.08	0.039	-0.036
6	1.25	0.232	0.071	0.044	-0.039
7	1.5	0.243	0.061	0.047	...

Click and scroll down to see more.

Plot the potential as a function of time: $T := F^{\langle 1 \rangle}$

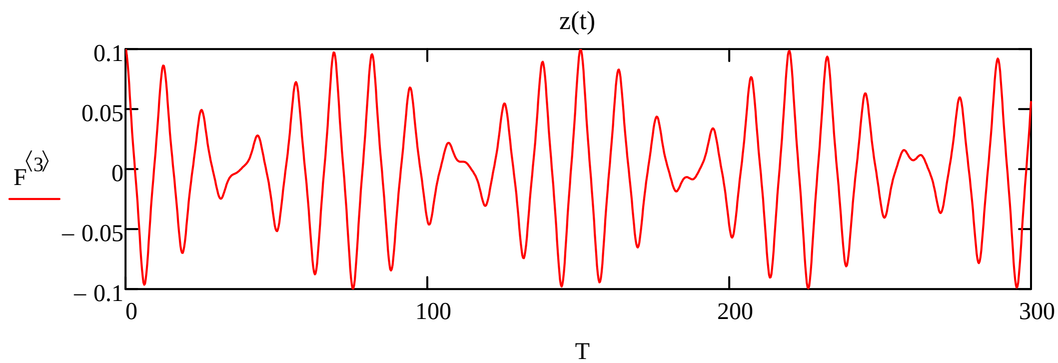


Particle radius as a function of time:



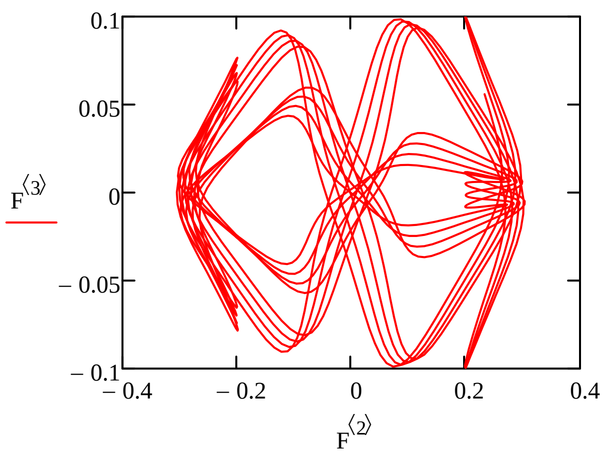
Notice above that the particle oscillates with a slow motion in the effective (ponderomotive) potential well made by the AC. On top of this motion is a fast "jitter" on the same time scale as the AC.

Plot z as a function of time:



The slow motion along the z axis occurs on a time scale that is shorter than that of the radial motion.

A projection of the motion onto the r,z plane:



This resembles the Lissajous patterns seen on oscillosopes.

Mathieu's equation can be written

$$\frac{d^2z}{dt^2} + \left[a - 2h\cos(2t)\right]z = 0$$

where the Ω has been removed by a change of variables. For our trap, the constant *a* = 0.
If there were an additional DC potential difference between the electrodes, *a* would be nonzero.
This equation is unstable for *a* = 0 if h < 0.908 which corresponds to (q/m) < 0.908 / 2 = 0.454 for our Paul trap.

Try it: raise q above from 0.45 to 0.48. What happens? Be sure to examine the vertical scales of the axes.
The particle must not move out of the trap before the sign of the force reverses. Will lowering the value of Ω make the trap more or less stable? **Try** increasing or decreasing Ω a factor of two and see what happens. Did you guess correctly?

Notes
1. There is a nice discussion of the Paul trap in H. Winter and H. W. Ortjohann, *American Journal of Physics* 59, 807 (1991). They describe how to build a Paul trap that stores dust particles in air.
2. The ideal potential variation ϕ(r,z) given on page 1 is obtained only if the electrodes are carfully machined to be hyperboloidal.