Numerical integration of derivatives by three methods

We will compare integration by (1) Euler's method, (2) the midpoint method, and (3) Runge Kutta. The function $4x^3$ will be integrated. The integral is x^4, so we will compare the results of the integration with x^4. We will integrate from x=0 to x=10 in 10 steps with $\Delta x=1$.

1. Euler's method (error proportional to Δx)

The slope of y(x) evaluated at x is used to find the value of y at (x+Δx):

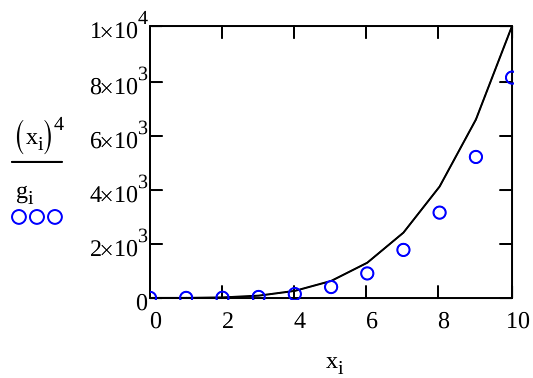
Euler's method uses: $y_n := y_{n-1} + \Delta x \cdot \frac{d}{dx} y(x_n)$

$h(x) := 4 \cdot x^3$ h(x) is the derivative of y

Define the grid: $imax := 10 \quad \Delta x := \frac{10}{imax} \quad i := 0 .. imax \quad x_i := i \cdot \Delta x$

$$g := \begin{array}{|l} g_{imax} \leftarrow 0 \\ \text{for } i \in 1 .. imax \\ \quad g_i \leftarrow g_{i-1} + \Delta x \cdot h(x_{i-1}) \\ g \end{array}$$

Initialize the answer vector g, then interate



$g_{imax} = 8100$ which is significantly less than 10,000.

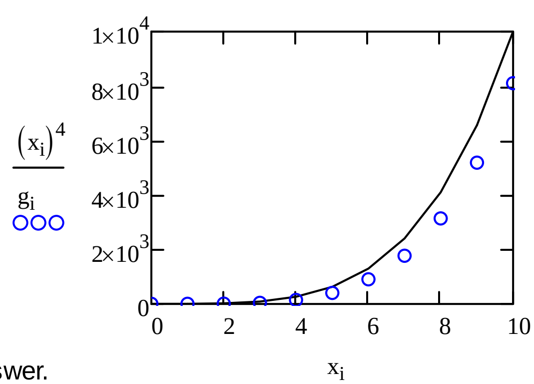
The points found by Euler's method lie signficantly below the curve x^4.

2. Midpoint method: (error proportional to (Δx)^2)

The midpoint method uses: $y_n := y_{n-1} + \Delta x \cdot \frac{d}{dx} y(x_n + 0.5 \cdot \Delta x)$

The derivative is evaluated half-way to the next point, and is "centered" on the interval of interest.

$$g := \begin{array}{|l} g_{imax} \leftarrow 0 \\ \text{for } i \in 1 .. imax \\ \quad g_i \leftarrow g_{i-1} + \Delta x \cdot h\left(x_{i-1} + \frac{\Delta x}{2}\right) \\ g \end{array}$$



$g_{imax} = 9950$

The midpoint method gives a more accurate answer.

3. Runge Kutta method: error proportional to (Δx)^4

The function rkfixed does the Runge Kutta integration starting at xstart and ending at xend using npoints as the number of intervals.

$xstart := 0$ starting x

$xend := 10$ ending x

$npoints := 10$ number of intervals, one less than the number of grid points.

$ystart := 0$ this is the starting value of y (the constant of integration)

$dy(x,y) := 4 \cdot x^3$ defines the derivative. The dy used by rkfixed MUST have two arguments.

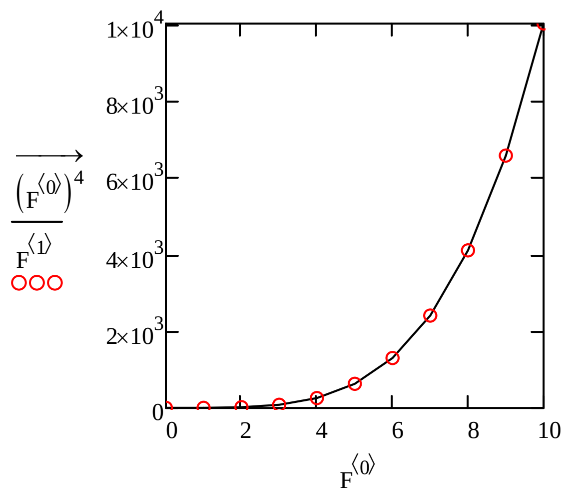
The function rkfixed does the integration and places the x and y values in the matrix F.

$F := rkfixed(ystart, xstart, xend, npoints, dy)$

x y

F =

	0	1
0	0.0000000	0.0000000
1	1.0000000	1.0000000
2	2.0000000	16.0000000
3	3.0000000	81.0000000
4	4.0000000	256.0000000
5	5.0000000	625.0000000
6	6.0000000	1296.0000000
7	7.0000000	2401.0000000
8	8.0000000	4096.0000000
9	9.0000000	6561.0000000
10	10.0000000	10000.0000000



$F_{imax,1} = 10000$ Runge Kutta has integrated this function without error.

Try it:

1. What happens in the first two methods if the number of points is doubled so that $\Delta x = 0.5$?

2. How far off is the integration by Euler's method if 100,000 points are used ($imax=10^5$) ?
Can you demonstrate that the error is proportional to Δx?