Numerical integration of a function

In this exercise we will use Simpson's rule to integrate a function and compare the answer with an analytic solution and to a numerical solution obtained by Mathcad.

We will integrate the function $y(x) := \frac{1}{1+x^2}$

We will make x_i a subscripted variable with 51 values: $imax := 50$

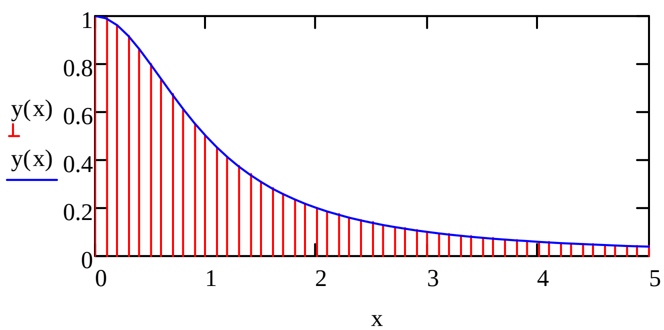
$i := 0 .. imax$ $\quad x_i := \left(\frac{i}{imax}\right)\cdot 5$ x goes from 0 to 5 by steps of 0.1.

Below is a plot of y(x). There is a stem plot and a line plot on one graph. The stem plot gives the vertical lines.

These are our x values:

$x =$

	0
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1
11	1.1
12	1.2
13	1.3
14	1.4
15	...



The analytic answer for the integral of y(x):

The definite integral of y(x) can be found using the **symbolic integrator** from the Evaluation menu:

$$\int_0^a \frac{1}{1+x^2}\,dx \rightarrow atan(a)$$

We will let a = 5 and use the arctangent function to find that:

$atan(5) = 1.373401$

Format | Result has been used to show six decimal places instead of the usual three.

The Simpson's rule answer:

The answer by Simpson's rule is the sum of the areas in all the trapezoids in the graph above.
The number of trapezoids is one less than the number of points.
The area of one trapezoid is the width of the base, dx, times the average height, or

$One\_Trapezoid\_Area := \frac{y(x_i) + y(x_{i+1})}{2}\cdot dx$ where $dx := x_2 - x_1$ $\quad dx = 0.1$

Evaluation of the equation above, which we will not use, has been turned off by right-clicking and selecting disable.

The area under the curve in the graph of y(x) above is:

$$\text{Area} := \sum_{i=0}^{\text{imax}-1} \left[\left(\frac{y(x_i) + y(x_{i+1})}{2}\right) \cdot dx\right] \qquad \text{Area} = 1.373388$$

Compare this to:

$$\text{atan}(5) = 1.373401$$

We can compare these two answers by finding their ratio:

$$\frac{\text{Area}}{\text{atan}(5)} = 0.999991 \quad \text{or} \quad \frac{\text{atan}(5)}{\text{Area}} = 1.000009$$

The numerical solution is very close to the analytic solution.

Mathcad's answer for the integral

Mathcad will give us a numerical answer if we simply type the integral:

$$\int_0^5 \frac{1}{1 + x^2}\, dx = 1.373400766955$$

which we can compare with $\text{atan}(5) = 1.373400766945$

Note that there is a difference in the last two decimal places.

We can ask for more accurate results from Mathcad by using the TOL command to set a smaller tolerance. Showing more digits with Format|Result is NOT the same as asking for more accuracy.

$\text{TOL} = 1 \times 10^{-3}$ This is what is used as the default value.

$\text{TOL} := 10^{-6}$ Now we have set TOL smaller, and will evaluate the integral again:

$$\int_0^5 \frac{1}{1 + x^2}\, dx = 1.373400766945$$

This new result agrees with what Mathcad tells us for atan(5). Mathcad's methods are hidden from us but are based upon Simpson's rule. Mathcad's method increases the number of trapezoids until the desired tolerance is obtained. You can find out more about Mathcad's methods by typing "integration" into the Mathcad help menu.

Simpson 's rule again:

What would we get if we used 5000 trapezoids instead of 50?

First we rewrite our equations so that when the number of trapezoids imax is changed, x_i and dx are changed as well. Let's divide the area into 5000 trapezoids:

$$\text{imax} := 5000 \qquad i := 0 .. \text{imax} \qquad x_i := \left(\frac{i}{\text{imax}}\right) \cdot 5 \qquad dx := x_2 - x_1 \qquad dx = 1 \times 10^{-3}$$

$$\text{Area} := dx \cdot \left[0.5 \cdot \left(y(x_0) + y(x_{\text{imax}})\right) + \sum_{i=1}^{\text{imax}-1} y(x_i)\right]$$

$$\text{atan}(5) = 1.3734007669$$

$$\text{Area} = 1.3734007657$$

Note that Area has been defined differently than at the top of this page. The new definition will **calculate much faster** because the multiplication by dx and divisions by 2 are not inside the sum and thus are done only a few times.

Try it: Increase or decrease the number of trapezoids, imax, and observe how the error changes. If you increase the number of trapezoids by a factor of 10, does the integration error decrease by a factor of 10? Can you see why Simpson's rule is called a "second order" method?