Solving a second order differential equation with Runge Kutta

Consider the equation F = ma. This is a second order equation that can be written

$$m\frac{d^2x}{dt^2} = F$$

Runge Kutta solves first order equations. We can solve the second order equation by turning it into two first order equations. A new variable is defined as the first time derivative of x.
We will call this v for historical reasons. The set of two equations below completely describes the problem:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{F}{m}$$

These are starting values: $x := 0$ $v := 0$ $F := 1$ $m := 1$

Runge Kutta can solve these two equations simultaneously.
We must define a vector with the two components x and v.

$$X := \begin{pmatrix} x \\ v \end{pmatrix}$$

On the right side I used the Insert|Matrix command from the matrix menu and selected one column and two rows. X is the vector containing the variables that are the answers.

Now define the derivative of the vector X

$$DX(t,X) := \begin{pmatrix} X_1 \\ \frac{F}{m} \end{pmatrix}$$

this says dx/dt = v = X_1

this says dv/dt = a = F/m

X_0 is x and X_1 is v. Mathcad numbers the components of vectors beginning with zero, unless you change it. For example, if you prefer to number beginning with 1, you would type ORIGIN:=1.
But this is NOT recommended except for the simplest programs, because the ORIGIN statement applies to everything you type after you use it.

Now solve the equations using Runge Kutta for t = 0 to t = 10. Let: $tstart := 0$ $tend := 10$

This interval will be divided into 10 parts: $npoints := 10$

$$Xstart := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We must put the starting values of x and v into the vector Xstart.
These are the constants of integration.

The equations of motion will be integrated by Runge-Kutta by using the **rkfixed** function.
The arguments are the starting value of x, the starting and ending values of t, the number of points on the time interval

$$F := rkfixed(Xstart, tstart, tend, npoints, DX)$$

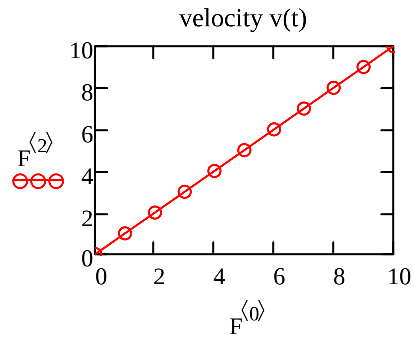
The answer matrix F will contain the times and the successive values of the vector X.

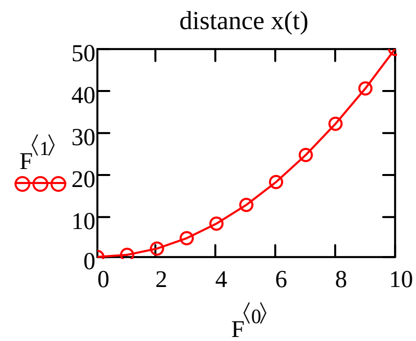
The answer matrix returned b y rkfixed:

Plots of the distance and velocity:

F =

	t	X_0	X_1
	0	1	2
0	0	0	0
1	1	0.5	1
2	2	2	2
3	3	4.5	3
4	4	8	4
5	5	12.5	5
6	6	18	6
7	7	24.5	7
8	8	32	8
9	9	40.5	9
10	10	50	10





These plots are what we expected.
With constant force, velocity increases linearly with time and distance increases with the square of time. Note that 0.5 a t^2 = 50 when t = 10.

Now throw a ball in two dimensions:

The vector X will contain four components: x, y, dx/dt, dy/dt.

$$Xstart := \begin{pmatrix} 0 \\ 0 \\ 50 \\ 50 \end{pmatrix}$$

x is X_0
y is X_1
dx/dt is X_2
dydt is X_3

My starting conditions are that I throw the ball from x,y = 0,0 with v = 50 in the x direction and v = 50 in the y direction.

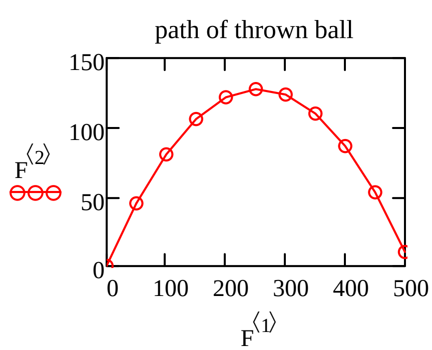
$g := 9.8$ is the acceleration of gravity, in the -y direction

$$DX(t, X) := \begin{pmatrix} X_2 \\ X_3 \\ 0 \\ -g \end{pmatrix}$$

dx/dt = v_x = X_2
dy/dt = v_y = X_3
dv_x/dt = 0
dv_y/dt = -g

We will again integrate from t = 0 to t = 10 using 10 points.

$F := \text{rkfixed}(Xstart, tstart, tend, npoints, DX)$



This is a plot of y versus x, which is the second column of F versus the first column. This indeed looks like what happens when you throw a ball.
The zero column of F, not used here, contains time.