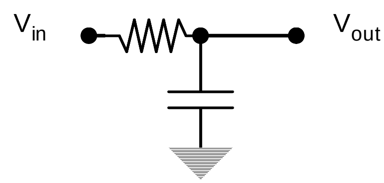
Circuit Analysis with Complex Variables

Simple circuits can easily be analyzed in a Mathcad spreadsheet. The impedances of circuits with resistive, inductive, and capacitive circuit elements can be expressed as complex numbers. These complex impedances can be used to analyze the response of the circuit as a function of frequency.

I. RC low pass filter:



For the filter circuit shown at right, R = 10 kΩ and C = 0.1 μF.

$$R_1 := 10\cdot 10^3\cdot\Omega$$

$C_1 := 0.1\cdot\mu F$ Mathcad knows the meaning of Ω and μF.

The corresponding impedances are: $Z_1(\omega) := R_1$ $\quad Z_2(\omega) := \frac{1}{i\cdot C_1\cdot\omega}$

The complex "i" has been entered by typing 1i, with NO space or character between the 1 and the i.

The time scale of interest is the RC time constant: $R_1\cdot C_1 = 1\times 10^{-3}\,s$

This means that the circuit gets interesting when ω = 1/RC.
Mathcad know that RC has units of seconds.

Define a range of ω going from 0.01 to 10 times the basic frequency scale of 1/RC.

$$\Omega_{RC} := \frac{1}{R_1\cdot C_1} \qquad \Omega_{RC} = 1\times 10^3\,\frac{rad}{s} \qquad \omega := 0.01\cdot\Omega_{RC}, 0.02\cdot\Omega_{RC} .. 10\cdot\Omega_{RC}$$

NOTE: You must explicitly type in the units of rad/s or else Mathcad will erroneously interpret this as Hz (cycles/s). You could divide Ω by 2π to correctly get the frequency in Hz. Alternatively, you can omit units entirely.

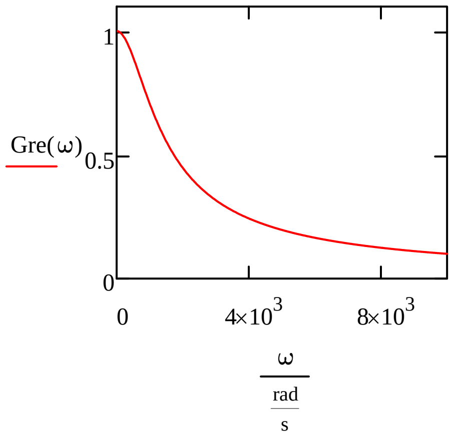
$$\frac{\Omega_{RC}}{2\cdot\pi} = 159.155\,\frac{1}{s}$$

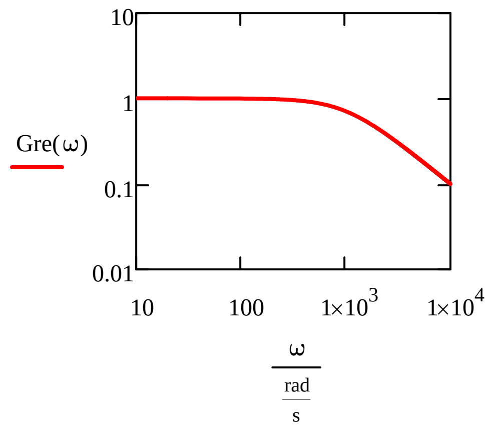
Our RC filter can be treated as a voltage divider with impedances $Z_1(\omega)$ and $Z_2(\omega)$,
The voltage division ratio (gain Gre) is:

$$Gre(\omega) := \frac{\left|Z_2(\omega)\right|}{\left|Z_1(\omega) + Z_2(\omega)\right|}$$

In this definition, the "re" means that we have taken the absolute values of the impedances so that G is real. This definition will return the amplitude, but not the phase, of the signal passed through the filter.

Here is the gain vs. frequency on a linear plot and a log-log plot. We can plot the frequency in units of rad/s by putting ω / (rad/sec) on the x-axis.





Where is the 3 db point?

The filter response is reduced by 3 db when the output signal is 0.707 of the input value.

We can check this using the definition: $db := 20\cdot\log(\sqrt{0.5})$ $db = -3.01$

We can find this point for our RC filter using the root finder (or 'zero finder').

The root finder requires a guess. We will guess something near 1/RC: $\omega_0 := 1500\cdot\frac{rad}{s}$

ω0 is used here because we have previously defined ω as a subscripted variable to use in the plots.

The root finding expression that we need is:

$$\mathrm{root}\left(\mathrm{Gre}(\omega_0) - \sqrt{0.5}, \omega_0\right) = 1\times10^3\,\frac{rad}{s}$$

As we expect, the root is the same as: $\frac{1}{R_1\cdot C_1} = 1\times10^3\,\frac{rad}{s}$

Phase angle

To determine the phase angle of V_{out} relative to V_{in}, we must use the complex value for the gain G.

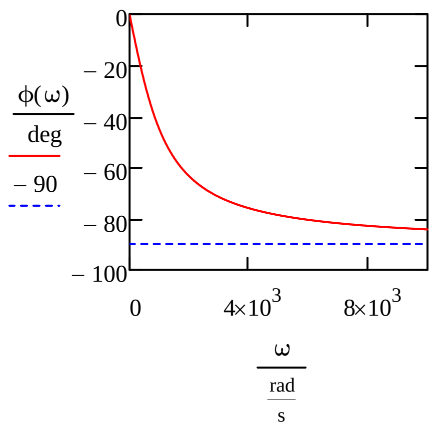
$$\mathrm{Gim}(\omega) := \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)}$$

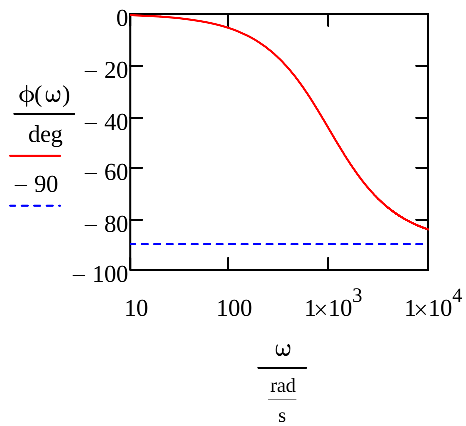
In this definition of G, the "im" means that we have used imaginary impedances and that G can have both real and imaginary parts.

In Mathcad, the phase angle of a complex number is found using the arg(z) function. The answer is given in radians unless we specify that we want the units to be degrees. We can plot something "in degrees" by dividing that quantity by deg on the y axis.

$\phi(\omega) := \arg(\mathrm{Gim}(\omega))$

Here are the linear and semi logarithmic phase angle vs. frequency plots:

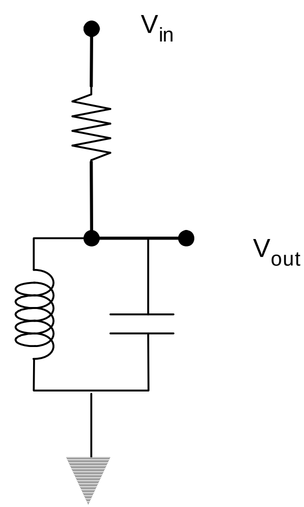




Try it: Why didn't we use a log-log plot?

II. RLC bandpass filter

The circuit at right is an RLC filter which has a response that is peaked at the resonant frequency. For this circuit, we can apply the same type of analysis that is used for the RLC circuit.



The circuit element values are:

$R_2 := 10 \cdot k\Omega$

$L_2 := 10 \cdot mH$

$C_2 := .01 \cdot \mu F$

The LC part of the circuit has a resonant frequency:

$\omega_0 := \dfrac{1}{\sqrt{L_2 \cdot C_2}}$ $\quad \omega_0 = 1 \times 10^5 \dfrac{rad}{s}$ $\quad \dfrac{\omega_0}{2 \cdot \pi} = 15.915\,kHz$

We will define a range of frequencies for plotting that go from 0.01 to 10 times ω0.

$\omega := 0.01 \cdot \omega_0, 0.02 \cdot \omega_0 .. 10 \cdot \omega_0$

The RLC circuit can be treated as a voltage divider with complex impedances.

The first impedance is simply: $Z_1(\omega) := R_2$

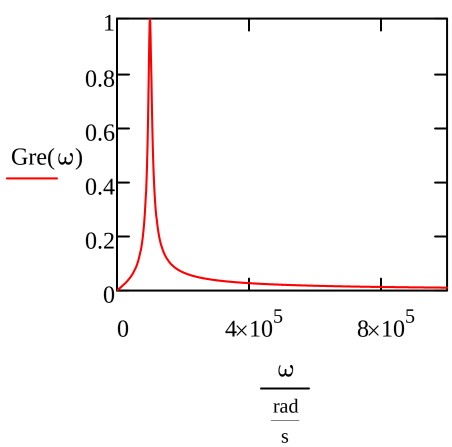
And the second impedance is the parallel combination of L and C:

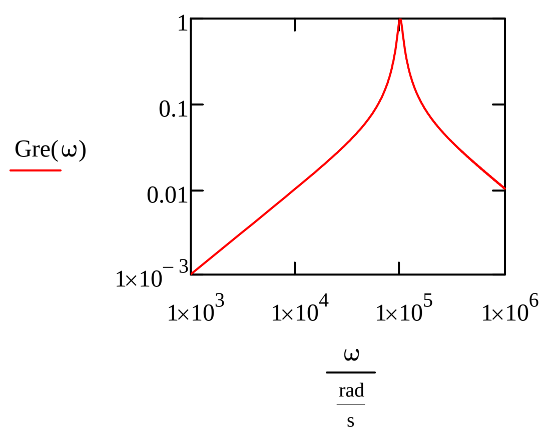
$$Z_2(\omega) := \frac{1}{\frac{1}{\frac{1}{i \cdot C_2 \cdot \omega}} + \left(\frac{1}{i \cdot L_2 \cdot \omega}\right)}$$

The gain and phase functions must be redefined with the new impedances:

$$Gre(\omega) := \frac{|Z_2(\omega)|}{|Z_1(\omega) + Z_2(\omega)|} \qquad Gim(\omega) := \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)} \qquad \phi(\omega) := \arg(Gim(\omega))$$

Here is the gain vs. frequency on a linear plot and a log-log plot:





Where are the 3 db points?

We see that the filter passes frequences near the resonant frequency and has a very peaked response. We can find the lower and upper 3 db points by providing the root finder with guesses that are too low and too high, respectively. .

The root that we need solves: Gre(ω) = 0.707.

The low guess is : $\omega_0 := 90000 \cdot \frac{rad}{s}$

The low root is: $\mathrm{root}(Gre(\omega_0) - 0.707, \omega_0) = 9.512 \times 10^4 \frac{rad}{s}$

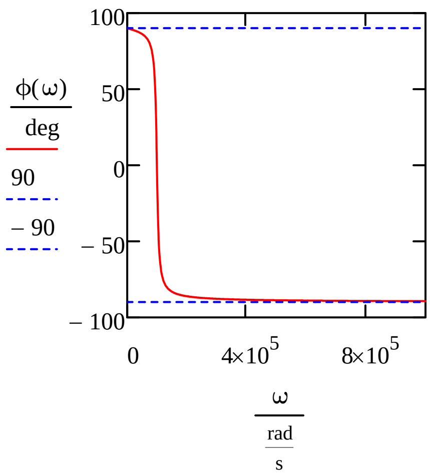
The high guess is : $\omega_0 := 110000 \cdot \frac{rad}{s}$

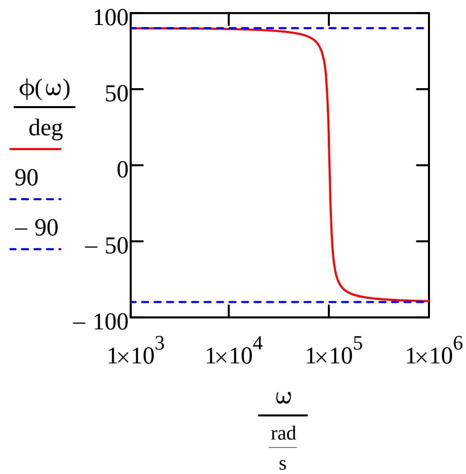
The high root is: $root(Gre(\omega_0) - 0.707, \omega_0) = 1.051 \times 10^5 \frac{rad}{s}$

We see that the half-bandwidth of ω is about 5 kHz.

Phase angle

Here are the linear and semi logarithmic phase angle vs. frequency plots:





For low frequencies, the inductive impedance dominates the parallel combination of L and C, and the resulting RL circuit has a "leading" phase of 90 degrees. For high frequencies, the capacitive impedance dominates the parallel L and C, and the circuit becomes analogous to the RC filter analyzed in the the first section of this exercise that has a "lagging" phase of 90 degrees at high frequencies.

Try it: Is the filter more or less peaked if R is doubled? Halved? Do the 3 db points move closer to one another or away from one another?