# Response of an RC filter to a square wave:

The experimenter often uses a square wave for timing and tiggering purposes. What happens to a square wave after passing through a low-pass filter? In this exercise we will express the square wave as the sum of a sine wave at the fundamental frequency and waves at the higher harmonics. Then we will use a complex circuit response function that attenuates and phase shifts each harmonic to find the new set of attenuated waves. The attenuated waves are then summed to find the waveform that comes out of the low-pass filter. The first example is a simple low-pass RC filter, the second example is a high-pass filter, and the third example is a RLC bandpass filter.

## *I. Constructing the square wave*

The Fourier decomposition of the square wave can be found in mathematical tables.
The frequencies and their amplitudes are given below.

The number of frequencies that we will consider is: $nmax := 32$ $n := 0, 1 .. nmax$

The fundamental frequency **f** that we will use is 1 kHz and the corresponding radian frequency is: $\Omega_{fund} := 2 \cdot \pi \cdot 1000$

The fundamental frequency and its odd harmonics are: $\Omega_n := (2 \cdot n + 1)\Omega_{fund}$

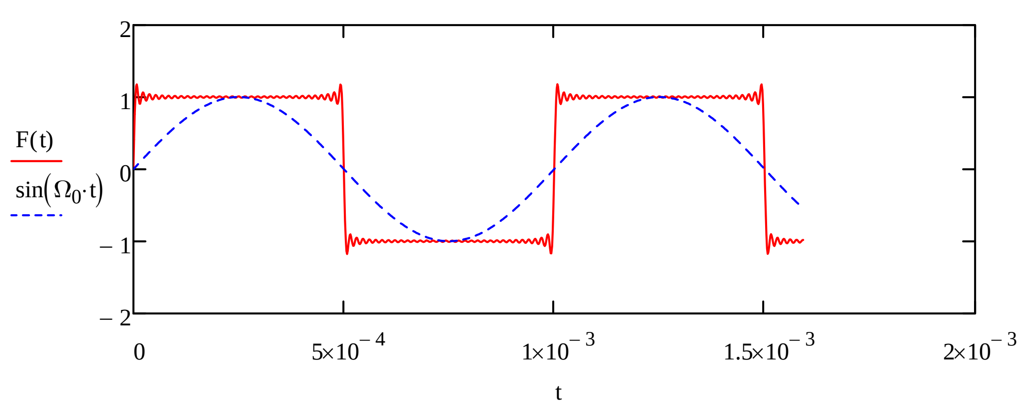
For the square wave, the amplitudes of the harmonics are: $a_n := \frac{2}{\pi} \cdot \left(\frac{2}{2n+1}\right)$
The square wave is then:

$$F(t) := \mathrm{Im}\left[\sum_{n=0}^{nmax} \left(a_n \cdot \exp\left(i \cdot \Omega_n \cdot t\right)\right)\right]$$

The imaginary part is used because we want to select the sine (not the cosine) and its harmonics.

The time scale of the problem is the inverse of the frequency $\Omega_{fund}$.
This time scale is used to define a range of times for plotting: $t := 0, \frac{0.01}{\Omega_{fund}} .. \frac{10}{\Omega_{fund}}$

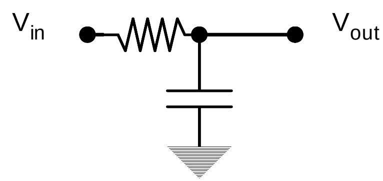
Plot of the sum representing the square wave



The plot of the square wave was made using 1000 points on the time axis. If fewer points had been used, the details in the square wave (at the "corners") would not have been plotted.

**Try it:** What happens if the number of harmonics (nmax) in the sum is halved? Doubled?

## *II. Square wave into an RC low pass filter*




Our RC filter is shown in the diagram at the right. The values of the circuit components are R = 10 kΩ and C = 0.05 μF.

$R := 1000$ ohms $\qquad C := 0.05 \cdot 10^{-6}$ Farads

The corresponding impedances are: $\qquad Z_1(\omega) := R \qquad Z_2(\omega) := \dfrac{1}{i \cdot C \cdot \omega}$

The time scale of interest is the RC time constant: $\qquad R \cdot C = 5 \times 10^{-5}$

The 3 db point of this filter is: $\qquad \omega_{3db} := \dfrac{1}{R \cdot C} \qquad \omega_{3db} = 2 \times 10^{4}$

Thus our filter easily passes the fundamental: $\qquad \Omega_{fund} = 6.283 \times 10^{3}$

For a voltage divider with impedances $Z_1(\omega)$ and $Z_2(\omega)$, the voltage division ratio (gain Gcx) is:

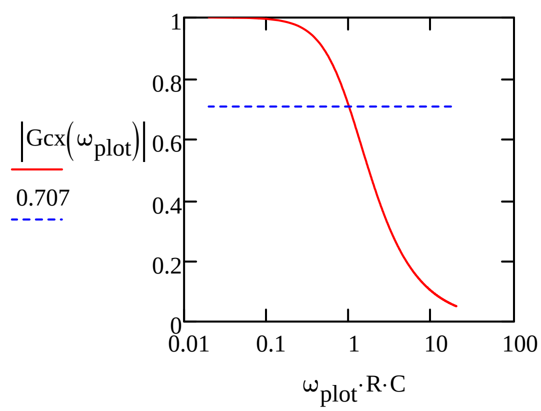
$$Gcx(\omega) := \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)}$$

The "cx" indicates that we have used a complex impedance to get the complex circuit response.

A range of frequencies for plotting is: $\qquad \omega_{plot} := 0, \dfrac{0.02}{R \cdot C} .. \dfrac{20}{R \cdot C}$

The absolute value of the response function of the filter is:

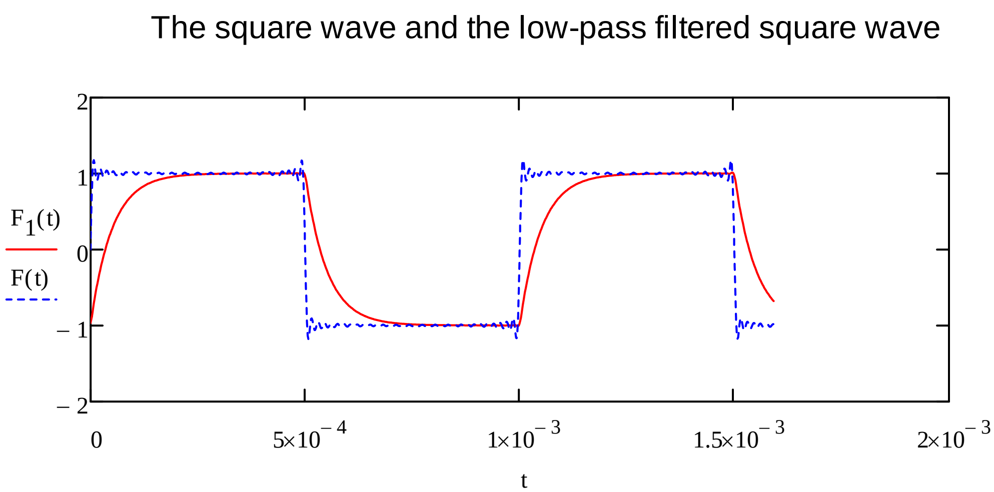
Note that we have plotted the horizontal scale in dimensionless form by multiplying by RC.



The filtered waveform is found by multiplying the amplitudes of the fundamental and each harmonic by the complex response function Gcx(ω) and then summing their contributions:

$$F_1(t) := \mathrm{Im}\left[\sum_{n=0}^{nmax} \left(a_n \cdot Gcx(\Omega_n) \cdot \exp(i \cdot \Omega_n \cdot t)\right)\right]$$

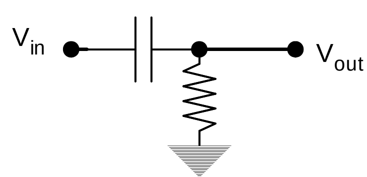
Note that the response function $Gcx(\Omega_n)$ goes inside of the summation since it depends on n.




Note that the filter has rounded the some (but not all) of the "corners" on the square wave. Thus the low pass filter doesn't just "smooth" the curve. The effect of the filter is more complex.

**Try it:** How is the filtered wave changed if R is doubled? Halved?

## *III. Square wave into an RC high pass filter*

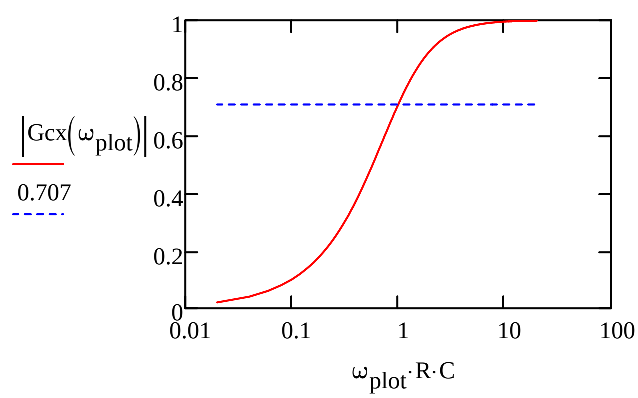



The high pass filter is really easy to do at this point because all that is necessary is to replace $Z_1$ with $Z_2$ and vice versa.

$$Gcx(\omega) := \frac{Z_1(\omega)}{Z_1(\omega) + Z_2(\omega)}$$

Note that the numerator is $Z_1$ and not $Z_2$.

The absolute value of the response function of the filter is:

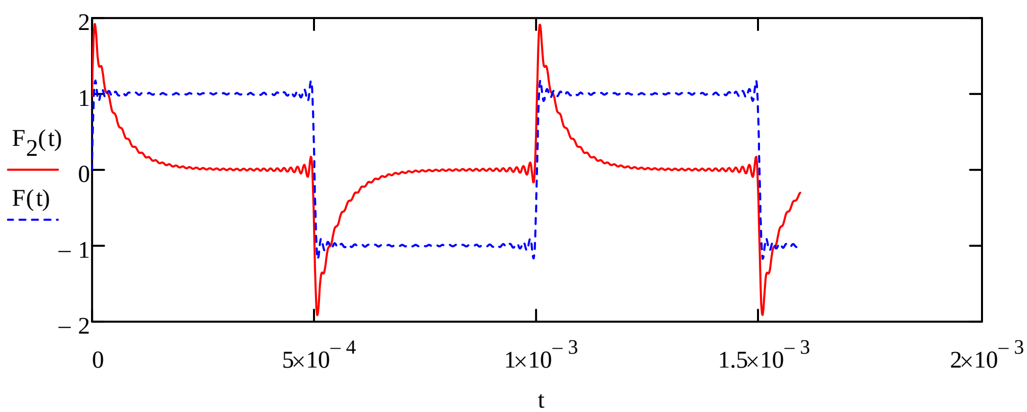



Again the horizontal axis has been made dimensionless.

The filtered waveform is found by multiplying the amplitudes of the fundamental and each harmonic by the complex response function Gcx(ω) and then summing their contributions:

$$F_2(t) := \mathrm{Im}\left[\sum_{n=0}^{nmax}\left(a_n \cdot Gcx(\Omega_n)\cdot \exp(i\cdot\Omega_n\cdot t)\right)\right]$$

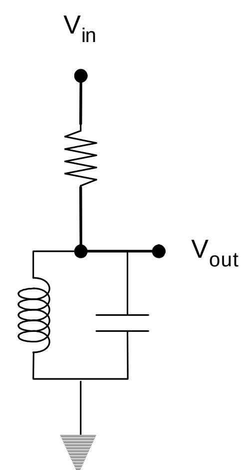
The square wave and the high-pass filtered square wave




**Try it:** Why is the amplitude 2.0 for this wave and 1.0 for the square wave ("trick" question)?

## *IV. RLC bandpass filter*

The circuit at right is an RLC filter which has a response that is peaked at the resonant frequency. For this circuit, we can apply the same type of analysis that is used for the RC circuit.




The circuit element values are:

$R := 10\cdot 10^{3}$ 10 KΩ

$L := 10\cdot 10^{-3}$ 10 milli Henries

$C := 0.01\cdot 10^{-6}$ 0.01 micro Farads

The LC part of the circuit has a resonant frequency:

$\omega_0 := \frac{1}{\sqrt{C\cdot L}}$ $\omega_0 = 1\times 10^{5}$ This frequency is much higher than the fundamental frequency of our square wave.

The RLC circuit can be treated as a voltage divider with complex impedances.

The first impedance is simply:

$$Z_1(\omega) := R$$

And the second impedance is the parallel combination of L and C:

$$Z_2(\omega) := \frac{1}{\left(\frac{1}{\frac{1}{i \cdot C \cdot \omega}}\right) + \left(\frac{1}{i \cdot L \cdot \omega}\right)}$$

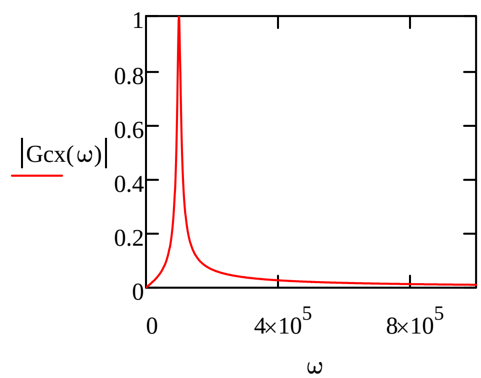
The gain that we defined above must be redefined with the new impedances:

$$Gcx(\omega) := \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)}$$

We will define a range of frequencies for plotting that go from 0.01 to 10 times $\omega_0$.

$$\omega := 0.01 \cdot \omega_0, 0.02 \cdot \omega_0 .. 10 \cdot \omega_0$$

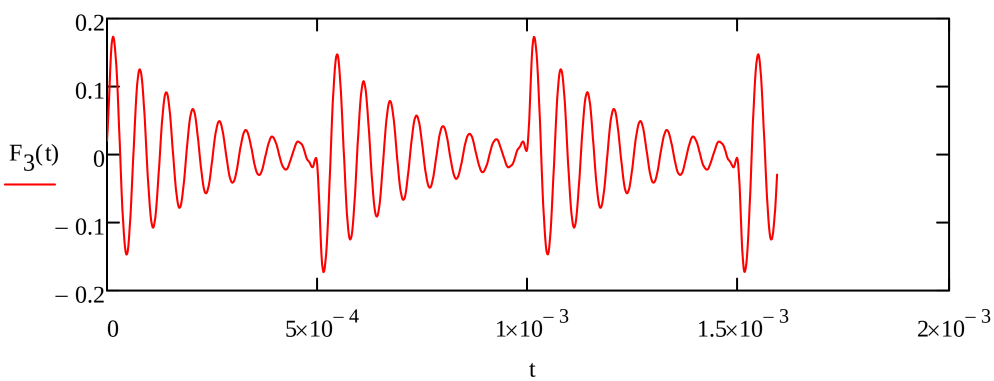
The absolute value of the complex response function has a peak at the resonant frequency of the LC part of the circuit:




The square wave after passage through the filter is:

$$F_3(t) := \mathrm{Im}\left[\sum_{n=0}^{nmax} \left(a_n \cdot Gcx(\Omega_n) \cdot \exp(i \cdot \Omega_n \cdot t)\right)\right]$$

The RLC filtered square wave




The plot shows that the transitions ("edges") of the square wave cause the LC part of the circuit to oscillate at the resonant frequency. The oscillations are damped by the resistor.

**Try it:** Add the square wave F(t) to the above graph and compare the amplitudes of the filtered and unfiltered waves.

**Try it:** Are the oscillations in the plot more or less heavily damped if the value of R is doubled?

**Try it:** Are the number of fast oscillations in one period of the sine wave equal to the number that you expect from comparing $\omega_0$ with $\Omega_{fund}$?

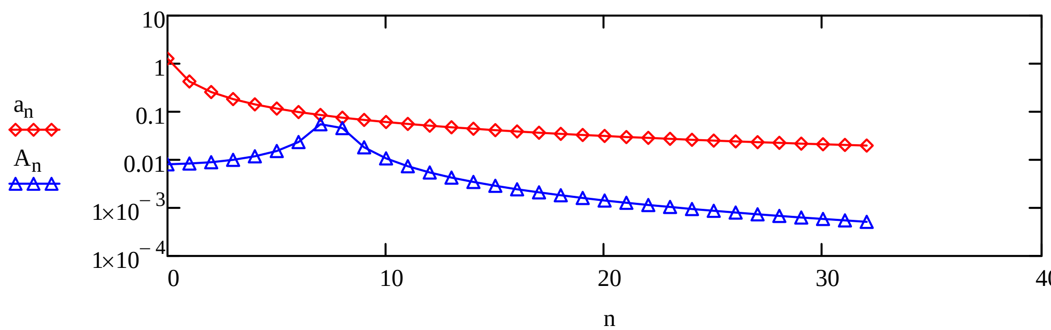
## *V. Frequency spectrum of the RLC filtered square wave*

The RLC filtered square wave appears to have a large contribution from the harmonic nearest the resonant frequency of the LC part of the circuit. Let's check this idea by plotting the amplitudes $A_n$ of the harmonics that are in the filtered wave along with the amplitudes $a_n$ in the unfiltered wave.

First, find the filtered amplitudes using the absolute value of the function Gcx to obtain a real number:

$$A_n := a_n \cdot \left| Gcx\left(\Omega_n\right) \right|$$

Amplitudes of the harmonics with and without RLC filtering:




The graph shows that the 7th and 8th harmonics of the fundamental are passed through the filter with nearly the full amplitude and the other harmonics are more attenuated:

$\Omega_7 = 9.425 \times 10^4$ $\Omega_8 = 1.068 \times 10^5$ These are near to: $\omega_0 = 1 \times 10^5$