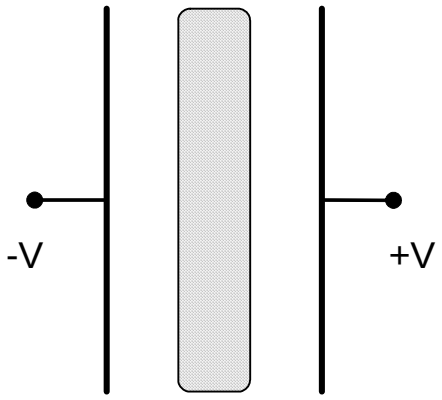
Laplace's Equation with a Dielectric

If a dielectric slab is placed between the plates of a capacitor the electric field is different from what it would be without the slab. We will assume that the thickness of the slab is less than the distance between the plates. The relaxation method introduced in the **Laplace's Equation** exercise will be used to find the distribution of the potential using a one-dimensional approximation, which is accurate if the plate separation is much smaller than the transverse dimensions. The exercise will then be extended to two dimensions with the infinite slab replaced by a dielectric rod.



Capacitor with dielectric slab.

When there is a dielectric and no free charges, it is the displacement D that has no divergence: This is Laplace's equation.

$$\vec{\nabla} \cdot \vec{D} = 0$$

When the dielectric constant ε is a scalar quantity, the displacement is related to the electric field by:

$$\vec{D} = \varepsilon \vec{E}$$

The electric field E is related to the potential Φ by:

$$\vec{E} = -\vec{\nabla}\Phi$$

Laplace's equation in one dimension

The relaxation method will find Φ given the boundary conditions +V and -V.
We will derive the mathematical relations to put into a **program loop**.
In one dimension, the gradient and divergence are simply derivatives with respect to x.
Let x_j be the j^{th} grid point and Φ_j and ε_j are abbreviations for $\Phi(x_j)$ and $\varepsilon(x_j)$.
The equation for D in finite difference form is:

$$D_j = \varepsilon_j E_j$$

If Δx is the spacing between grid points, the derivative of D evaluated at the j^{th} point is:

$$\left[\frac{dD}{dx}\right]_j = \frac{D_{j+1/2} - D_{j-1/2}}{\Delta x} = \frac{1}{\Delta x}\left[\varepsilon_{j+1/2} E_{j+1/2} - \varepsilon_{j-1/2} E_{j-1/2}\right] = 0$$

Above we have introduced a "half grid" defined by the relation:
The derivative is centered at the point x_j and is thus accurate to second order.

$$x_{j+1/2} = x_j + \tfrac{1}{2}\Delta x$$

Expressing E as the negative gradient of Φ, Laplace's equation becomes:

$$\frac{1}{(\Delta x)^2}\left[\varepsilon_{j+1/2}\left(\Phi_{j+1}-\Phi_j\right)-\varepsilon_{j-1/2}\left(\Phi_j-\Phi_{j-1}\right)\right]=0$$

This relation requires that we know ε half way between grid points. We can either introduce a new grid of half-points or we can use the usual grid for ε, assume that ε varies smoothly, and approximate ε_j by the average of the values at the neighboring grid points. The second method is more convenient.

$\varepsilon_{j+1/2}=\frac{\varepsilon_j+\varepsilon_{j+1}}{2}$ With this approximation, Laplace's equation becomes:

$$\left(\varepsilon_j+\varepsilon_{j+1}\right)\left(\Phi_{j+1}-\Phi_j\right)-\left(\varepsilon_{j-1}+\varepsilon_j\right)\left(\Phi_j-\Phi_{j-1}\right)=0 \quad \text{or}$$

$$\left(\varepsilon_j+\varepsilon_{j+1}\right)\Phi_{j+1}+\left(\varepsilon_{j-1}+\varepsilon_j\right)\Phi_{j-1}-\left(\varepsilon_{j-1}+2\varepsilon_j+\varepsilon_{j+1}\right)\Phi_j=0 \quad \text{or}$$

$$\Phi_j=\frac{\left(\varepsilon_j+\varepsilon_{j+1}\right)\Phi_{n+1}+\left(\varepsilon_{j-1}+\varepsilon_j\right)\Phi_{j-1}}{\left(\varepsilon_{j-1}+2\varepsilon_j+\varepsilon_{j+1}\right)}$$

This last expression will be used in a program loop to find Φ_j by successive approximations. The left side of the equation defines the new value of Φ_j in terms of the old Φ_{j-1} and Φ_{j+1}. The new Φ values not on the boundary are simply weighted averages of the old values.

The one-dimensional problem (part 1)

Assume that the capacitor plates have a separation of 12 units and that the thickness of the slab is 6 units. The slab will be assumed to have a dielectric constant of ε = 3. Outside the slab ε = 1. Laplace's equation in one dimension requires that D be constant. Suppose D = 1, then E = 1 outside the slab and E = 0.33 inside the slab. The integral of E from the left plate to the right plate is 8, so this is the potential difference. Then Φ changes by 8 units and for boundary conditions on Φ we can use -4 and +4.

The slab thickness: $xmax := 12$ There will be jmax+1 grid points: $jmax := 48$

The grid point separation will be: $\Delta x := \frac{xmax}{jmax}$ $\Delta x = 0.25$

The grid points will be defined as: $j := 0 .. jmax$ $x_j := j \cdot \Delta x$ Then: $x_{jmax} = 12$

The starting guess for Φ_j for the intial iteration will have the boundary conditions applied at j = 0 and j = jmax, and will simply be zero everywhere else:

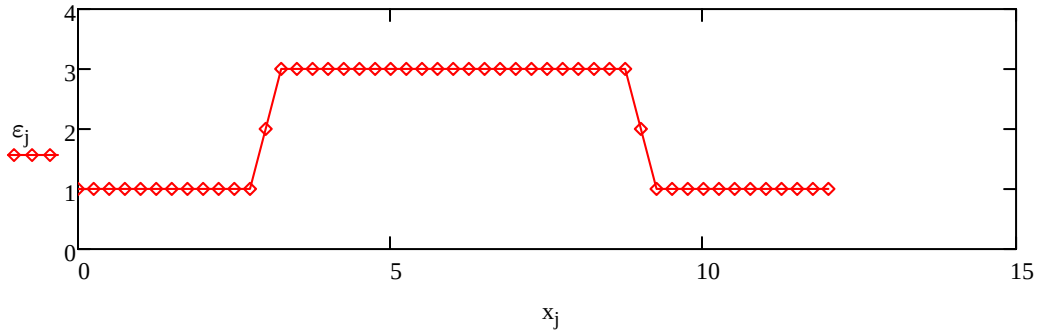
$PhiStart_j := 0$ $PhiStart_0 := -4$ $PhiStart_{jmax} := 4$

The values for ε_j will be defined so that the slab with $\varepsilon = 3$ is centered between the plates:

$\varepsilon_j := 1$ $\quad$ $jslab := 12 .. 36$ $\quad$ $\varepsilon_{jslab} := 3$ $\quad$ $\varepsilon = 3$ on a subset of points jj.

The dielectric constant changes from $\varepsilon = 1$ to $\varepsilon = 3$ precisely at points 16 and 32. These two points are "half" in the dielectric and "half" in the vacuum. The accuracy of our method is improved if we assign an ε value to these points that is the average of the values on either side:

$\varepsilon_{12} := \frac{1+3}{2}$ $\quad$ $\varepsilon_{36} := \frac{1+3}{2}$ $\quad$ **A plot** of the ε values is:



The program loop

Number of iterations. $iter := \frac{jmax^2}{4}$ iter scales with jmax^2 because the iterative procedure is similar to the procedure for a diffusion problem.

A matrix $Phi_{i,J}$ will hold the successive approximations. The row i holds the ith iteration.

$Phi :=$ $Phi_{iter,jmax} \leftarrow 0$

for $j \in 0 .. jmax$

$Phi_{0,j} \leftarrow PhiStart_j$

for $i \in 1 .. iter$

for $j \in 1 .. jmax - 1$

$$Avg_j \leftarrow \frac{\left(\varepsilon_j + \varepsilon_{j+1}\right)\cdot Phi_{i-1,j+1} + \left(\varepsilon_j + \varepsilon_{j-1}\right)\cdot Phi_{i-1,j-1}}{\varepsilon_{j-1} + 2\cdot\varepsilon_j + \varepsilon_{j+1}}$$

for $j \in 1 .. jmax - 1$

$Phi_{i,j} \leftarrow Avg_j$

$Phi_{i,0} \leftarrow Phi_{i-1,0}$

$Phi_{i,jmax} \leftarrow Phi_{i-1,jmax}$

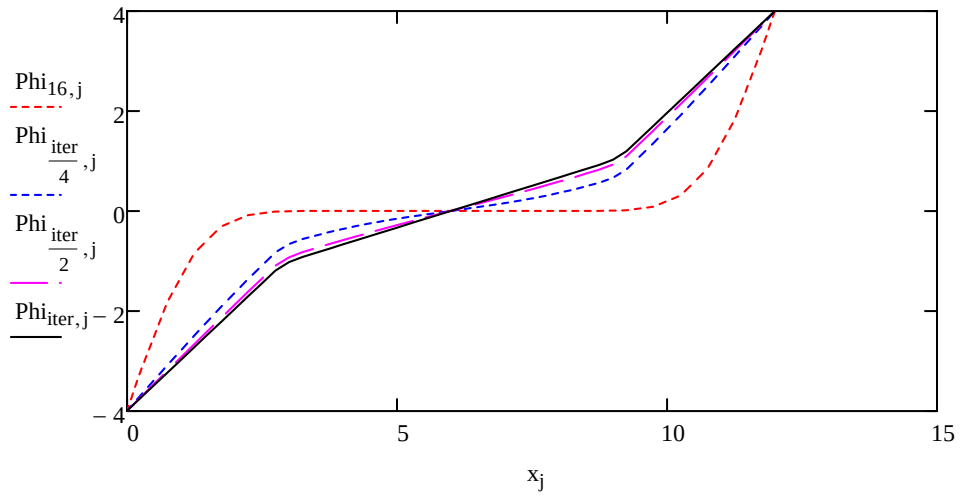
Phi

This is the matrix Phi containing the successive approximations:

Phi =

	0	1	2	3	4	5
0	-4	0	0	0	0	0
1	-4	-2	0	0	0	0
2	-4	-2	-1	0	0	0
3	-4	-2.5	-1	-0.5	0	0
4	-4	-2.5	-1.5	-0.5	-0.25	0
5	-4	-2.75	-1.5	-0.875	-0.25	...

This is a **plot** of the final iteration and several intermediate iterations:



We see that the plots are converging toward the solution.

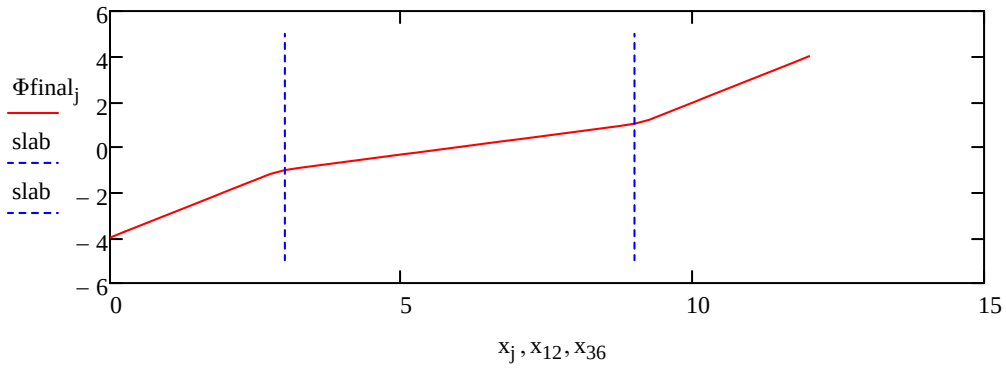
Put the final iteration into a vector Φfinal for easier plotting:

$$\Phi final := \left(Phi^{T}\right)^{\langle iter \rangle}$$

A range of values for plotting:

$$slab := -5, -4.9 .. 5$$

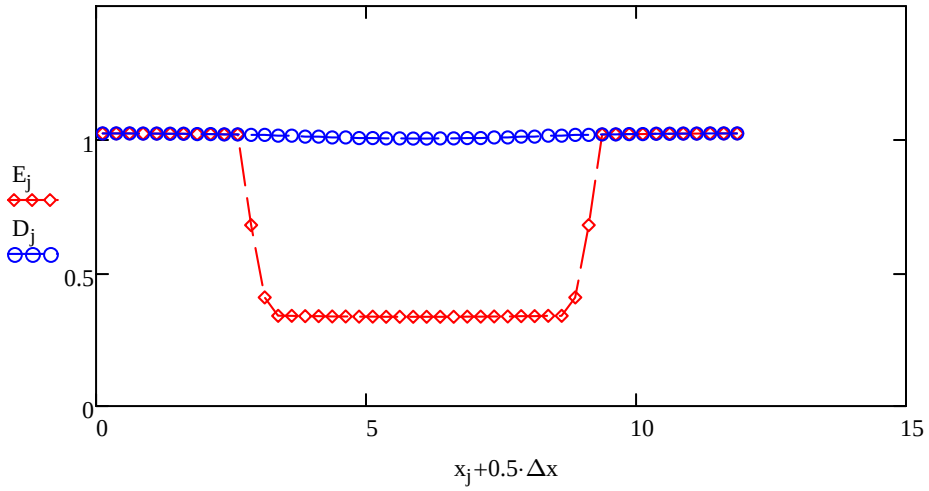
Plot Φfinal again showing the boundaries of the slab:



Find values for E using finite differencing of Φ: $j := 0 .. jmax - 1$ $E_j := \frac{\Phi final_{j+1} - \Phi final_j}{\Delta x}$

Note that E_j is centered at $x_{j+1/2}$.

Plot of values of E and D = ε E: $D_j := \left(\frac{\varepsilon_{j+1} + \varepsilon_j}{2}\right) \cdot E_j$



The problem was set up initially so that E = 1 outside of the dielectric and E = 0.33 inside of the dielectric and this is approximately what has been found.

A typical value inside the slab: $E_{\frac{jmax}{2}} = 0.334$

A typical value outside the slab: $E_4 = 1.021$

Dielectric in two dimensions (part 2)

Now apply what has been learned above to a similar problem in two dimensions using Cartesian coordinates.

$\vec{\nabla} \cdot \vec{D} = 0$ in two dimensions becomes: $+\frac{d}{dx}\left[\varepsilon \frac{d}{dx}\Phi\right] + \frac{d}{dy}\left[\varepsilon \frac{d}{dy}\Phi\right] = 0$

There will be two grids: x_j and y_k. Both Φ and ε will be dependent upon both variables.

Recall that in one dimension we obtained:

$$\left(\varepsilon_j + \varepsilon_{j+1}\right)\Phi_{j+1} + \left(\varepsilon_{j-1} + \varepsilon_j\right)\Phi_{j-1} - \left(\varepsilon_{j-1} + 2\varepsilon_j + \varepsilon_{j+1}\right)\Phi_j = 0$$

In two dimensions, Laplace's equation becomes

$$\frac{1}{(\Delta x)^2}\left[\left(\varepsilon_{j,k}+\varepsilon_{j,k+1}\right)\Phi_{j,k+1}+\left(\varepsilon_{j,k-1}+\varepsilon_{j,k}\right)\Phi_{j,k-1}-\left(\varepsilon_{j,k-1}+2\varepsilon_{j,k}+\varepsilon_{j,k+1}\right)\Phi_{j,k}\right]+$$
$$\frac{1}{(\Delta y)^2}\left[\left(\varepsilon_{j,k}+\varepsilon_{j+1,k}\right)\Phi_{j+1,k}+\left(\varepsilon_{j-1,k}+\varepsilon_{j,k}\right)\Phi_{j-1,k}-\left(\varepsilon_{j-1,k}+2\varepsilon_{j,k}+\varepsilon_{j+1,k}\right)\Phi_{j,k}\right]=0$$

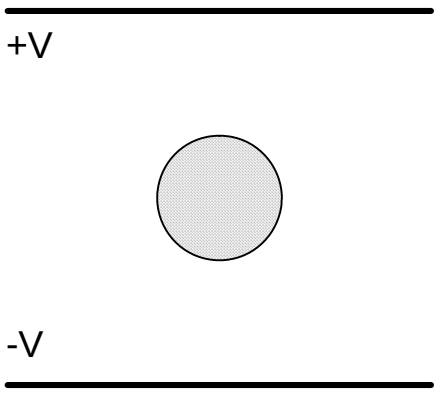
This can be rearranged to give us the equation for $\Phi_{j,k}$ to use in the iterations:

$$\Phi_{j,k}=\frac{\left(\varepsilon_{j,k}+\varepsilon_{j,k+1}\right)\Phi_{j,k+1}+\left(\varepsilon_{j,k-1}+\varepsilon_{j,k}\right)\Phi_{j,k-1}+\left(\varepsilon_{j,k}+\varepsilon_{j+1,k}\right)\Phi_{j+1,k}+\left(\varepsilon_{j-1,k}+\varepsilon_{j,k}\right)\Phi_{j-1,k}}{\left(\varepsilon_{j-1,k}+2\varepsilon_{j,k}+\varepsilon_{j+1,k}\right)+\left(\varepsilon_{j,k-1}+2\varepsilon_{j,k}+\varepsilon_{j,k+1}\right)}$$

where we have assumed that $\Delta x = \Delta y$ for simplicity.

Define the problem

The domain will be 12 units on each side with Δx and Δy both equal to 0.25. We will place an infinitely long cylindrical rod in the domain oriented so that the axis of the rod is out of the page. The radius of the rod will be 2 units and the relative dielectric constant will be 3. The applied potential difference will be 12 units.



Periodic boundary conditions for x

On the left and right side of the domain we will apply periodic boundary conditions. This means that to the left and right of the domain there are indentical domains. If kmax is the subscript for the right boundary, this means that $\Phi_{kmax-1} = \Phi_{kmax+1}$ by symmetry. At the left boundary, $\Phi_{-1} = \Phi_1$. We cannot easily use -1 for an index so we will shift the k values so that k = 1 is the left boundary and k = 0 is the point to the left of the left boundary. With these definitions, the periodic boundary conditions are applied if the we the formula above for $\Phi_{j,k}$ and remember to place the value of $\Phi_{j,2}$ into $\Phi_{j,0}$ and place the value of $\Phi_{j,kmax-1}$ into $\Phi_{j,kmax+1}$. These assignments will go at the end of the program loop.
If the boundary conditions are periodic, the Φ values on the left and right boundaries are maxima or minima. This means that the electric field E_x at the left and right boundaries are zero and that periodic boundary conditions are the same as zero electric field boundary conditions.

The x grid points

Our subscript for the x coordinate will extend from 0 to 26 to allow for the additional two points.

$xmax := 12$ The width of the domain.

$\Delta x = 0.25$ The same as before.

$jmax := \frac{xmax}{\Delta x}$ $jmax = 48$ kmax+1 is the number of points in the domain.

$j := 0 .. jmax + 2$ There are jmax+3 points (rather than jmax+1) in the grid because of the special image points at j = 0 and j = jmax+2.

$x_j := j \cdot \Delta x - \Delta x$ This definition includes the extra points.

$x_0 = -0.25$ $x_{jmax+2} = 12.25$ These are the extra two points needed to enforce periodic boundary conditions.

The y grid points:

We will use a square domain and have $\Delta x = \Delta y$. A potential will be applied at the upper and lower boundaries as though the dielectric were between horizontal capacitor plates.

$ymax := xmax$ $ymax = 12$ The height of the domain.

$\Delta y := \Delta x$ We assumed this earlier in our derivation.

$kmax := \frac{ymax}{\Delta y}$ $kmax = 48$

$k := 0 .. kmax$ $y_k := k \cdot \Delta x$ $rows(y) = 49$ Defines the y grid.

The ε matrix

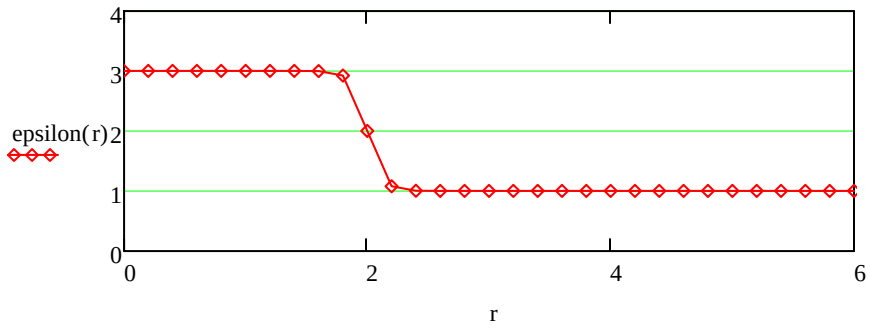
The value of ε is unity outside the dielectric and is 3 inside of the dielectric, which has a radius of 2 units. The center of the grid is at (x,y) = (6,6).

$\varepsilon_{j,k} := 1$ Initialize the ε grid. $rows(\varepsilon) = 51$ $cols(\varepsilon) = 49$

ε is 3 at the center of the rod and drops to 1 outside the rod. We will have a more accurate solution if on the boundary of the dielectric we assign a value of ε that is the average of the value on either side. We can have a smooth change from 3 to 1 if we use the **tanh** function:

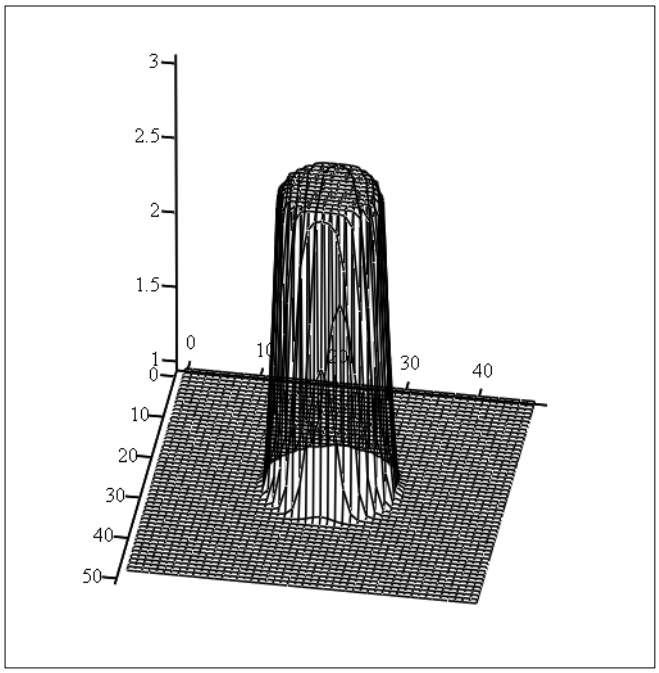
$epsilon(r) := 2 - 1 \cdot \tanh\left(\frac{r - 2}{0.5 \cdot \Delta x}\right)$ The denominator of the argument was chosen to be half the grid spacing.

A plot of epsilon(r): $r := 0, 0.2 .. 6$ Values of r for plotting.



$\varepsilon_{j,k} := \text{epsilon}\left[\sqrt{(x_j - 6)^2 + (y_k - 6)^2}\right]$ This places the values of the function on the matrix for ε. Note that the circular rod is centered at (6,6).

The spatial dependence of ε can be visualized using a **contour plot**:



ε

The y boundary conditions

We will assume a potential of +6 at the top boundary and -6 at the bottom boundary. These conditions will be enforced in the program loop by keeping the $\Phi_{j,k}$ values constant for j = 0 and j = jmax.

The initial guess for $\Phi_{j,k}$

The initial guess will go in a matrix PhiGuess and this will have the boundary conditions applied. The guess will be zero except at the top and bottom boundaries.

$\text{PhiGuess}_{j,k} := 0$ $\text{PhiGuess}_{j,0} := -6$ $\text{PhiGuess}_{j,kmax} := 6$

The program loop

$\text{iter} := \frac{jmax^2}{4}$ This is the number of iterations.

Recall that the subscript j is for the x direction and goes to jmax +2
and the subscript k for the y direction goes to kmax.
The program loop is not done for k = 0 and k = kmax because we do not need to alter the fixed boundaries. The points at j = 0 and j = jmax + 2 are defined using the symmetry condition.
In this two-dimensional problem, we do not save the intermediate iterations.

Phi :=
- $\text{Phi} \leftarrow \text{PhiGuess}$
- for $i \in 0 .. \text{iter}$
  - for $k \in 1 .. kmax - 1$
    - for $j \in 1 .. jmax + 1$
      - $\text{Avg}_{j,k} \leftarrow \frac{\left(\varepsilon_{j-1,k} + \varepsilon_{j,k}\right)\cdot \text{Phi}_{j-1,k} + \left(\varepsilon_{j,k} + \varepsilon_{j+1,k}\right)\cdot \text{Phi}_{j+1,k} \ldots + \left(\varepsilon_{j,k} + \varepsilon_{j,k-1}\right)\cdot \text{Phi}_{j,k-1} + \left(\varepsilon_{j,k} + \varepsilon_{j,k+1}\right)\cdot \text{Phi}_{j,k+1}}{\varepsilon_{j-1,k} + \varepsilon_{j+1,k} + 4\cdot\varepsilon_{j,k} + \varepsilon_{j,k+1} + \varepsilon_{j,k-1}}$
  - for $k \in 1 .. kmax - 1$
    - $\text{Phi}_{0,k} \leftarrow \text{Phi}_{2,k}$
    - $\text{Phi}_{jmax+2,k} \leftarrow \text{Phi}_{jmax,k}$
    - for $j \in 1 .. jmax + 1$
      - $\text{Phi}_{j,k} \leftarrow \text{Avg}_{j,k}$
- Phi

Note that a line break (... +) has been incorporated into the definition of Avg by typing Ctrl + Enter. The first "for j, for k" loops of the program loop applies the iteration scheme derived earlier, The variable Avg is the weighted average of the Φ values that surround the point j,k :

$$\Phi_{j,k} = \frac{\left(\varepsilon_{j,k} + \varepsilon_{j,k+1}\right)\Phi_{j,k+1} + \left(\varepsilon_{j,k-1} + \varepsilon_{j,k}\right)\Phi_{j,k-1} + \left(\varepsilon_{j,k} + \varepsilon_{j+1,k}\right)\Phi_{j+1,k} + \left(\varepsilon_{j-1,k} + \varepsilon_{j,k}\right)\Phi_{j-1,k}}{\left(\varepsilon_{j-1,k} + 2\varepsilon_{j,k} + \varepsilon_{j+1,k}\right) + \left(\varepsilon_{j,k-1} + 2\varepsilon_{j,k} + \varepsilon_{j,k+1}\right)}$$

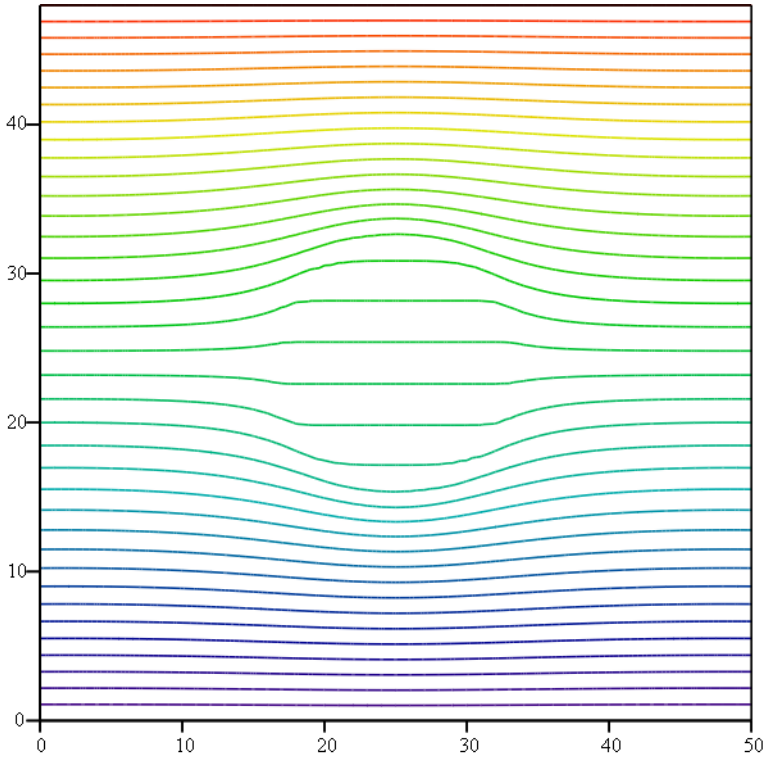
The second "for j, for k" loops applies the periodic boundary conditions at j = 0 and j = jmax. Then all the other Φ values are updated in this loop using the weighted average Avg.

This is part of the final array for $\Phi_{j,k}$:

Phi =

	0	1	2	3	4	5	6	7	8
0	-6	-5.702	-5.405	-5.11	-4.817	-4.526	-4.24	-3.958	-3.681
1	-6	-5.702	-5.406	-5.11	-4.817	-4.527	-4.241	-3.959	-3.682
2	-6	-5.702	-5.406	-5.11	-4.817	-4.527	-4.241	-3.959	-3.682
3	-6	-5.702	-5.405	-5.11	-4.817	-4.527	-4.24	-3.958	-3.681
4	-6	-5.702	-5.405	-5.109	-4.816	-4.526	-4.239	-3.957	-3.679
5	-6	-5.702	-5.404	-5.108	-4.815	-4.524	-4.237	-3.954	-3.676
6	-6	-5.701	-5.403	-5.107	-4.813	-4.522	-4.234	-3.951	-3.673
7	-6	-5.701	-5.402	-5.105	-4.811	-4.519	-4.231	-3.947	-3.668
8	-6	-5.7	-5.401	-5.103	-4.808	-4.516	-4.227	-3.942	...

The **contour plot** of the pontential shows the the potential gradient is weaker inside the dielectric:



Phi

Note that the contours are nearly straight within the dielectric rod.

The electric fields Ex and Ey are found by finite differencing. The results can be viewed as a vector plot. To reduce the number of vectors, the vectors will be plotted only for even numbered points.

The indices jj and kk will be used to find the fields at even numbered points

$$jj := 0 .. \frac{jmax}{2} - 1 \qquad kk := 0 .. \frac{kmax}{2} - 1$$

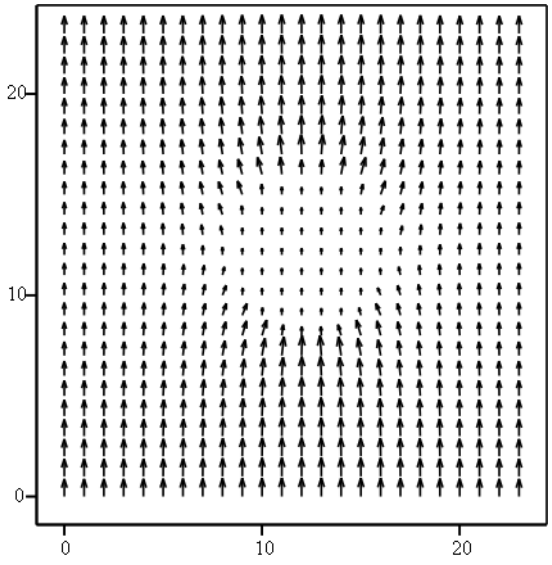
The x component of the E field is:

$$EX_{jj,kk} := \frac{Phi_{2\cdot jj+1,\,2\cdot kk} - Phi_{2\cdot jj,\,2\cdot kk}}{\Delta x}$$

The y component of the E field is:

$$EY_{jj,kk} := \frac{Phi_{2\cdot jj,\,2\cdot kk+1} - Phi_{2\cdot jj,\,2\cdot kk}}{\Delta y}$$

A plot of the electric field vectors:

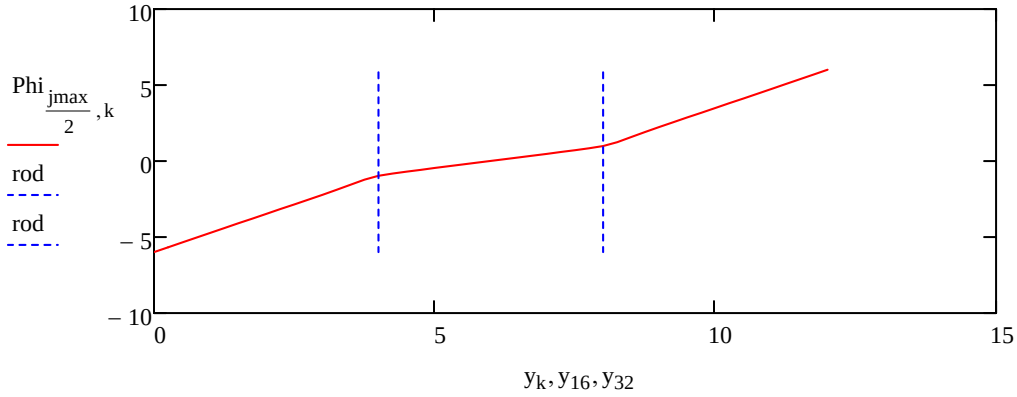


(EX, EY)

These z values are for the vertical lines representing the rod in the plot below:

$$rod := -6, -5.9 .. 6$$

This plot shows the potential values along the midplane:



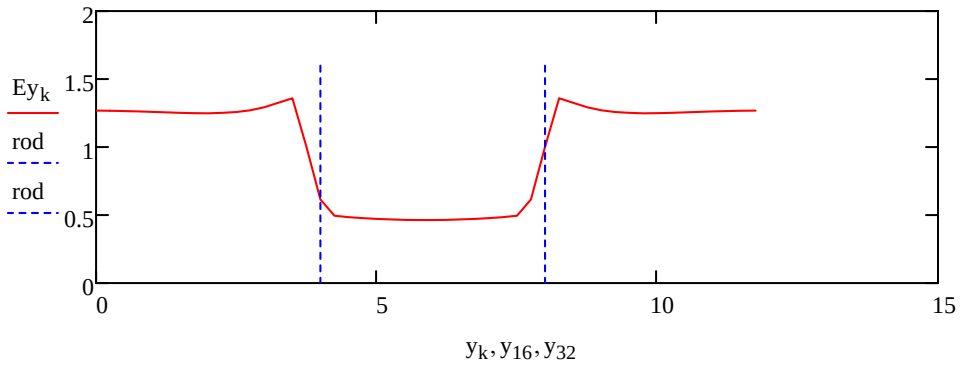
The y component of the E field values along the midplane are:

The j value at the middle of the x axis is: $jmid := \frac{jmax}{2}$

$k := 0 .. kmax - 1$ $\quad Ey_k := \frac{Phi_{jmid,k+1} - Phi_{jmid,k}}{\Delta y}$

Define a range of values to indicate the rod in the plot below: $rod := 0, 0.1 .. 1.6$

This plot is of the vertical component of the electric field at the middle value of x:



Compare this plot to the plot for the slab on page 4.

The analytic solution for the rod (see reference below) is a uniform E field inside the rod that is smaller by the factor 1/[1+ε/2]. That means that the field inside our rod with ε = 3 should be 0.4 of the undisturbed field. We applied a potential difference of 12 across a distance of 12 units which gives an undisturbed field of 1.0 and thus a predicted field inside the rod of 0.4. This is near to the computed field at the center of the rod, which is:

$Ey_{0.5kmax} = 0.462$

We do not expect nearly perfect agreement because the rod is far from the boundaries in the theory but is not far in the computational model. **Try it:** Is the E field within the rod closer to the analytic solution if the rod is smaller or if the empty space is larger?

Try it: Show that the solution has converged by doubling the number of iterations and finding

that the new solution is near to the old solution. First save part of the matrix Phi (on page 10) by cutting it and using **paste special** to save it as a bitmap that will not change. Then compare the numbers in the saved matrix to the new matrix found with the number of iterations doubled. Then save this Phi, double the number of iterations again and compare.

Notes

1) In this exercise we assumed that the permittivity of vacuum is unity in order to avoid scientific notation. This has no consequences because the value multiplies all terms in the equation. The ε for the dielectric is the relative value.

Reference

D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, New Jersey, 1981). The polarization of a dielectric rod is Problem 4.22.