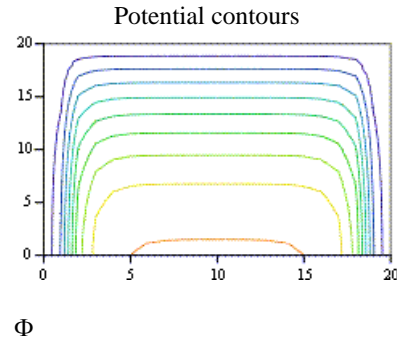


## Poisson's equation in cylindrical geometry: $\phi(r)$ and $\phi(r,z)$

We will solve, in cylindrical coordinates, Poisson's equation. First without  $z$  dependence and then with  $z$  dependence. In the first example, we consider an infinitely long cylinder of ions and we find the potential profile within the cylinder, assuming that the potential at the surface is zero. In the second example, the cylinder is of finite length and is inside a grounded cylindrical container. In the third example, the cylinder of charge has both a smaller length and a smaller radius than the cylindrical container.



First consider Poisson's equation in cylindrical coordinates with no  $z$  dependence:

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \phi(r) = \frac{\rho}{\epsilon_0}$$

This equation can be solved by the relaxation method. We begin by writing Poisson's equation in finite-difference form, and express the new value of  $\phi$  in terms of the neighboring values, using page 1 of the Debye Shielding Exercise as a guide:

$$\phi_j \leftarrow 0.5 \cdot \left( 1 + \frac{\Delta r}{2 \cdot r_j} \right) \cdot \phi_{j+1} + 0.5 \cdot \left( 1 - \frac{\Delta r}{2 \cdot r_j} \right) \cdot \phi_{j-1} + 0.5 \cdot \Delta r^2 \cdot \frac{\rho_j}{\epsilon_0}$$

At the origin,  $j = 0$ , we need to know  $\phi_{-1}$ , which is not defined, so we must use a different formula for  $j = 0$  than the one above. The point  $r = -1$  is the same point as the point at  $r = +1$ , because of the nature of cylindrical symmetry. We can use the formula for  $\phi_j$  above, but for the value of  $\phi_{-1}$ , we must substitute the value of  $\phi_1$ . Then for the special point at the origin, the

$$\phi_0 \leftarrow \phi_1 + 0.5 \cdot \Delta r^2 \cdot \frac{\rho_0}{\epsilon_0} \quad \text{Physical constants:} \quad \epsilon_0 := 8.854 \cdot 10^{-12} \quad q := 1.6 \cdot 10^{-19}$$

At the right boundary,  $\phi$  is constant and is always equal to the specified boundary condition. We will set up a grid of 25 points that spans 0 to 0.01 meters.

$j_{\max} := 25$	number of grid points along $r$	$j := 0 \dots j_{\max}$	index for grid points
$r_{\max} := 0.01$	the maximum radius in meters	$\Delta r := \frac{r_{\max}}{j_{\max}}$	the grid point spacing, in meters
$r_j := j \cdot \Delta r$	grid point locations		
BC := 0	the boundary condition applied at $r_{\max}$ . Note that for cylindrical coordinates we do not specify a boundary condition on the axis. In an experiment there is no metal boundary at $r = 0$ . The potential at $r = 0$ "floats."		
$n := 10^{15}$	the number density of ions, the units are $\text{m}^{-3}$ .		
$\rho_j := n \cdot q$	the charge density at the $j$ th grid point. We assume the same charge density $n$ at all points inside the cylinder of charge.		

### How do we know what to choose for the number of iterations?

Poisson's equation is related to the diffusion equation because both have two spatial derivatives. The answer "diffuses" across the grid points. The characteristic time for diffusion is the square of the distance. If you measure the distance in grid points, then a first guess for the number of iterations is the square of the number of grid points. In this case, 625. After finding an answer, you should double the number of iterations to see if the answers change significantly. If they do, keep doubling the number until the answers do not change significantly.

iters := 2000    k is the number of iterations. 2000 is about 3 times jmax squared.

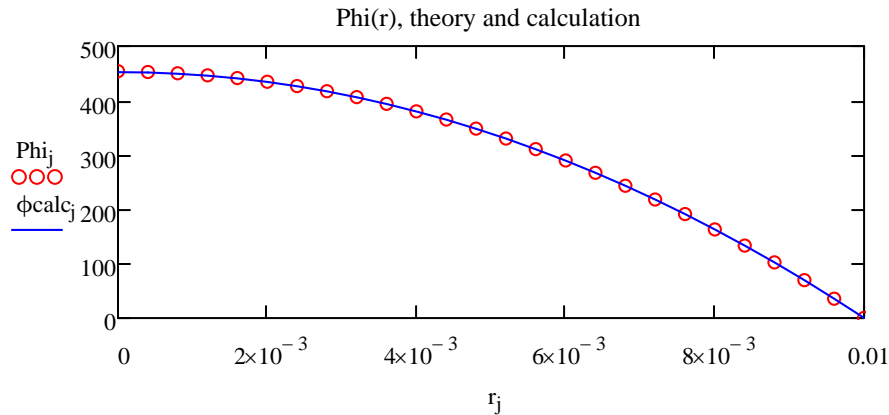
This is a program loop that solves our problem by successive approximations:

```
Phi :=
  for j ∈ 0 .. jmax
    | φj ← 0
    | φnew,j ← 0
    φjmax ← BC
    for k ∈ 0 .. iters
      | φnew,0 ← φ1 + 0.5 · Δr2 ·  $\frac{\rho_0}{\epsilon_0}$ 
      | for j ∈ 1 .. jmax - 1
          | φnew,j ← 0.5 ·  $\left(1 + \frac{\Delta r}{2 \cdot r_j}\right) \cdot \phi_{j+1} + 0.5 \cdot \left(1 - \frac{\Delta r}{2 \cdot r_j}\right) \cdot \phi_{j-1} + 0.5 \cdot \Delta r^2 \cdot \frac{\rho_j}{\epsilon_0}$ 
      | for j ∈ 0 .. jmax
          | φj ← φnew,j
    φ
```

$$\phi_{\text{calc},j} := \frac{n \cdot q}{4 \cdot \epsilon_0} \cdot \left[ (r_{\text{jmax}})^2 - (r_j)^2 \right]$$

This is the analytic answer for comparison.

This graph is a comparison of the analytic answer (the line) with the answer from successive approximations, the circles.



We can also check accuracy by comparing the values found on the axis:

$$\Phi_{i_0} = 453.598 \quad \phi_{calc_0} = 451.773 \quad \text{The calculated and analytic values are both near 452 volts.}$$

**Try it:** Does using a finer grid give a better answer? Does using more iterations?

### ***Poisson's equation in r,z coordinates***

The finite difference form of Poisson's equation becomes

$$\frac{1}{(\Delta r)^2} \left[ \left( 1 + \frac{\Delta r}{2r_j} \right) \phi_{i,j+1} - 2\phi_{i,j} + \left( 1 - \frac{\Delta r}{2r_j} \right) \phi_{i,j-1} \right] + \frac{1}{(\Delta z)^2} [\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}] + \frac{\rho_{i,j}}{\epsilon_0} = 0$$

which can be rewritten

$$\phi_{i,j} \leftarrow \frac{1}{2[1 + (\Delta r / \Delta z)^2]} \left[ \left( 1 + \frac{\Delta r}{2r_j} \right) \phi_{i,j+1} + \left( 1 - \frac{\Delta r}{2r_j} \right) \phi_{i,j-1} + \frac{(\Delta r)^2}{(\Delta z)^2} [\phi_{i+1,j} + \phi_{i-1,j}] + (\Delta r)^2 \frac{\rho_{i,j}}{\epsilon_0} \right]$$

We define a two-d grid with indices i and j.

$i_{max} := 20$        $j_{max} := 20$        $j_{max}$  should be an even number because  $j_{max}/2$  is used later.

$i := 0 .. i_{max}$       index for z grid points

$j := 0 .. j_{max}$       index for r grid points

$r_{max} := 0.01$       the maximum radius

$\Delta r := \frac{r_{max}}{j_{max}}$       the r grid point spacing       $\Delta r = 5 \times 10^{-4}$  meters

$z_{max} := 0.1$       the length of the z axis

$\Delta z := \frac{z_{\max}}{i_{\max}}$  the z grid point spacing  $\Delta z = 5 \times 10^{-3}$  meters

$r_j := j \cdot \Delta r$  the location of the r grid points

$i_{\text{ters}} := 1000$  this is the number of iterations that we will do

$n := 10^{15}$  the number density of ions  $\text{m}^{-3}$

### Boundary conditions: the rod of charge:

$\rho_{i,j} := 0$  Initialize the charge density to zero for all i and j.  
 $i_{\text{subset}} := 2 \dots i_{\max} - 2$  This makes a rod of charge that does not go all the way to the end of the cylindrical chamber. The rod of charge is spaced two grid points away from the end caps.  
 $\rho_{i_{\text{subset}},j} := n \cdot q$

### Boundary conditions: the cylindrical walls:

$BC_{i,j} := 1001$  This assigns 1001 to all grid points. We use 1001 to indicate places that are NOT boundary conditions. Next we will change the grid points at the boundary points to the correct values. Later we use an "if" statement to find which points are boundary points.

$BC_{i,j_{\max}} := 0$  The outer boundary has zero potential. This loop is done for all i values but only the largest j value.

$BC_{0,j} := 0$   $BC_{i_{\max},j} := 0$  The top and bottom end cap potentials are set to zero. All values of j are used but only i = 0 and i = i<sub>max</sub> are used at the boundaries..

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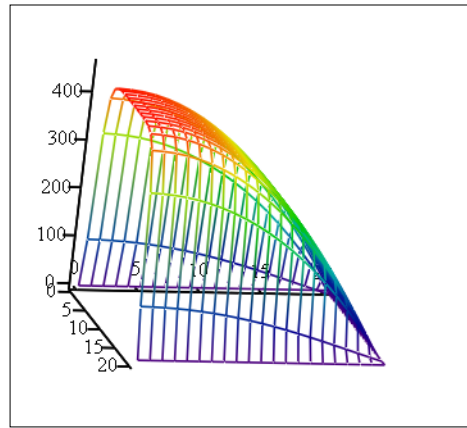
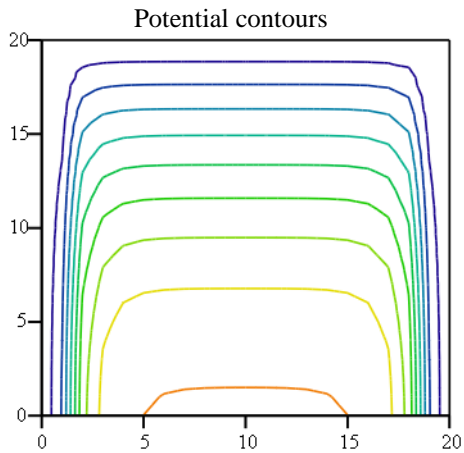
Phi2(ρ) := i ← 0
           for j ∈ 0..jmax
             for i ∈ 0..imax
               Phi2i,j ← if(BCi,j < 1000, BCi,j, 0)
             for k ∈ 0..iters
               for i ∈ 1..imax - 1
                 Ri,0 ←  $\frac{0.5}{1 + \left(\frac{\Delta r}{\Delta z}\right)^2} \left[ \frac{\Delta r^2}{\Delta z^2} \cdot (\text{Phi2}_{i-1,0} + \text{Phi2}_{i+1,0}) + 2 \cdot \text{Phi2}_{i,1} + \frac{\Delta r^2 \cdot \rho_{i,j}}{\epsilon_0} \right]$ 
                 for j ∈ 1..jmax - 1
                   Ri,j ←  $\frac{0.5}{1 + \left(\frac{\Delta r}{\Delta z}\right)^2} \left[ \frac{\Delta r^2}{\Delta z^2} \cdot (\text{Phi2}_{i-1,j} + \text{Phi2}_{i+1,j}) + \left(1 + \frac{\Delta r}{2 \cdot r_j}\right) \cdot \text{Phi2}_{i,j+1} \dots \right]$ 
                    $+ \left(1 - \frac{\Delta r}{2 \cdot r_j}\right) \cdot \text{Phi2}_{i,j-1} + \frac{\Delta r^2 \cdot \rho_{i,j}}{\epsilon_0}$ 
                 for i ∈ 1..imax
                   for j ∈ 0..jmax
                     Phi2i,j ← if(BCi,j < 1000, BCi,j, Ri,j)
               Phi2

```

In the program loop, the long formula has been displayed on two levels using shift-enter (notice the ...) which combines a line break with a plus sign.

The potential profile for a short cylinder of ions within a cylindrical boundary:

$\Phi := \text{Phi2}(\rho)$  We put the potential values in a matrix called  $\Phi$  for plotting. Phi2 has been made a function of  $\rho$  so we can easily recalculate Phi2 for a different charge distribution  $\rho$ .



$\Phi$

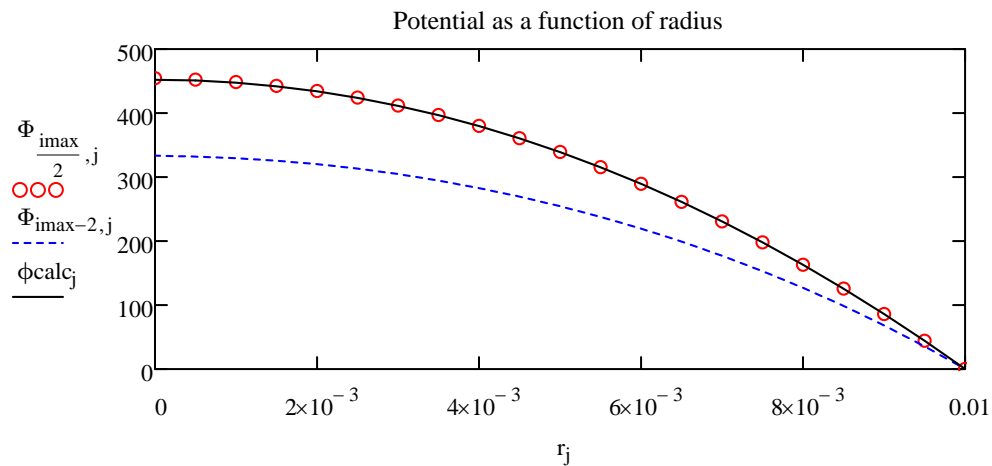
$\Phi$

**For the left graph, radius is the vertical axis and z is the horizontal axis.**

Note that the axes are labelled in grid points rather than cm or m.

$$\phi_{calc,j} := \frac{n \cdot q \cdot \left[ (r_{jmax})^2 - (r_j)^2 \right]}{4 \cdot \epsilon_0}$$

This is the analytic answer. In the figure below, the potential within the cylinder is close to the analytic answer near the middle of the cylinder,  $i = imax/2$ , but not near the end,  $i = imax-2$ .



In this plot the ion cylinder is 10 mm in radius and the cylindrical chamber is also 10 mm in radius. Suppose the cylinder of ions has a smaller radius than the cylinder that contains it.

**Next, the ions will fill only half the radius of the container.**

**Don't look! Answer this question:**

Will the potential in the vacuum between the cylinder of ions and the wall be zero?

Boundary conditions part 1 again, the smaller rod of charge:

$\rho_{i,j} := 0$  This definition is done for all values of  $i$  and  $j$ .

$i_{subset} := 2 .. i_{max} - 2$   $j_{outer} := 0 .. \frac{j_{max}}{2}$  The indices  $i_{subset}$  and  $j_{outer}$  define a rod of charge that is smaller than the cylindrical boundary in both the  $r$  and  $z$  directions. The rod of charge extends only half way radially to the outer boundary. The rod of charge is spaced two grid points away from the end caps.

$\rho_{i_{subset}, j_{outer}} := n \cdot q$

$\Phi := \text{Phi2}(\rho)$  This solves the problem again with the new ion charge cylinder that does not go all the way to the outer boundary. We don't have to write down the program loop again because we have made  $\text{Phi2}$  a function of the ion charge distribution  $\rho$ .

Note in the figure below that the maximum potential on axis is smaller than before because the radius of the charge cylinder is smaller. The charge on the rod, per unit length, is  $n q \pi (0.5 r_{max})^2$ . From Gauss's law, the radial field  $E$  at any radius outside the rod is  $n q (0.5 r_{max})^2 / 2r\epsilon_0$ . That means we can find the potential outside the rod by integrating  $E$  to get a potential that varies logarithmically:

$$\phi_{outside}(x) := \frac{n \cdot q \cdot (0.5 \cdot r_{max})^2}{2 \cdot \epsilon_0} \cdot (\ln(r_{max}) - \ln(x)) \quad j_{outer} := \frac{j_{max}}{2} .. j_{max}$$

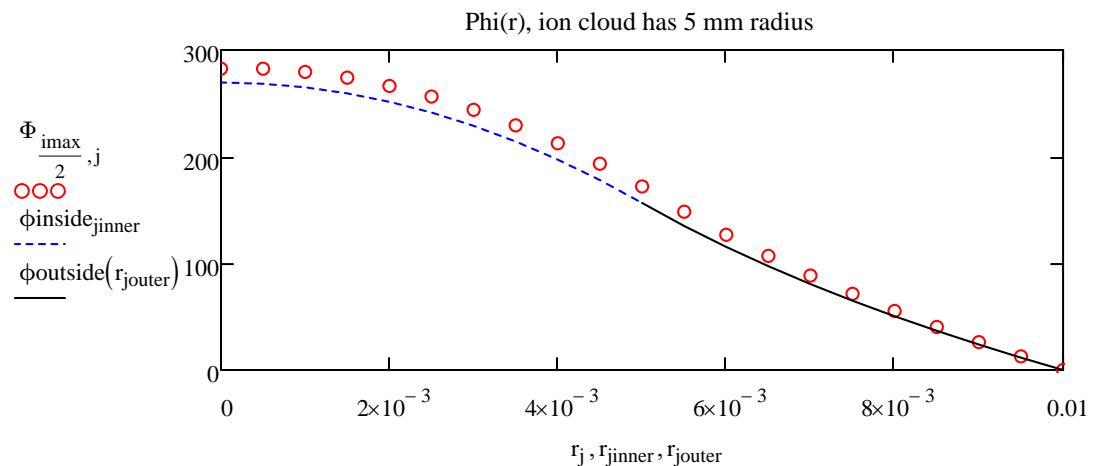
$j_{outer}$  gives values for  $r$  which are between the charge cylinder and the wall.

Inside the charge cylinder the potential varies parabolically as before. We must use the correct constant of integration, which means that we must add the solution inside the ion column to the solution outside.

$$\phi_{inside, j} := \frac{n \cdot q \cdot \left[ \left( \frac{r_{j_{max}}}{2} \right)^2 - (r_j)^2 \right]}{4 \cdot \epsilon_0} + \phi_{outside} \left( \frac{r_{max}}{2} \right) \quad j_{inner} := 0 .. \frac{j_{max}}{2}$$

$j_{inner}$  gives values for  $r$  which are inside the charge cylinder

This plot compares the analytic and numerical solutions.



In this plot the ion cylinder is 5 mm in radius and the cylindrical chamber is 10 mm in radius. The potential profile is parabolic inside the ion cylinder and logarithmic outside the ion cylinder.

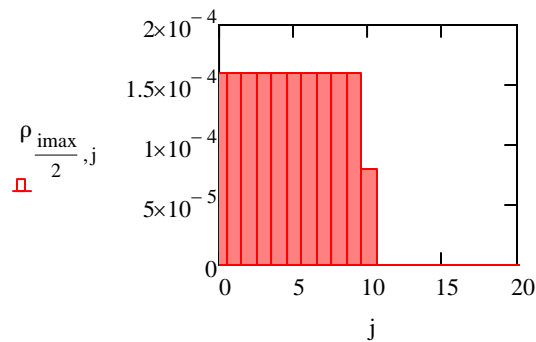
It appears as though our numerical answer above deviates from the analytic answer near the midplane. We have assigned a charge density of  $nq$  to the point  $r = r_{max}/2$  which is the outer boundary of the charge cylinder. To the right of this point the charge density is zero. If we average

the charge density to the right and to the left of this point, the charge density is  $nq/2$ . Let's make that change and see what happens.

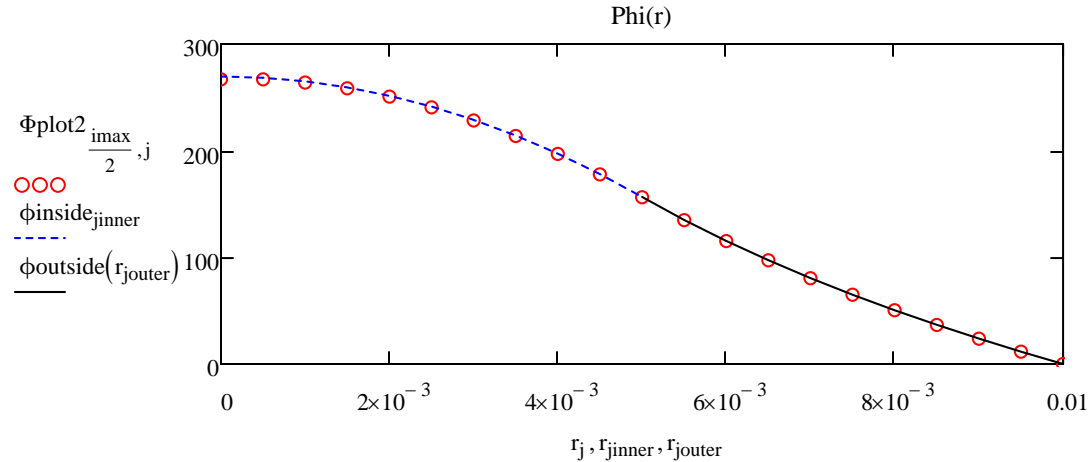
$\rho_{\text{isubset}, \frac{j_{\text{max}}}{2}} := \frac{n \cdot q}{2}$  At the outer boundary of the charge cylinder, we assign the average charge density of  $nq/2$ .

$\Phi_{\text{plot2}} := \text{Phi2}(\rho)$  Recalculate with the new boundary condition at the outer edge of the charge cylinder.

$j := 0 .. j_{\text{max}}$



This bar graph shows that the charge density at the grid points is  $nq$  for points inside the rod of charge and is  $nq/2$  at the point on the surface of the rod of charge. This plot is made at the center of the rod, at  $z$  location  $i_{\text{max}}/2$ .



$\phi_{\text{inside}}_0 = 269.516$

$\Phi_{\frac{i_{\text{max}}}{2}, 0} = 282.472$

The agreement between the analytic solution and the numerical solution is much better using the average value of charge density at the boundary.