# The ionosphere: Separation of ions by gravity

The ionosphere has hydrogen ions H$^+$ and oxygen ions O$^+$. The proportion of H$^+$ ions increases with altitude. This increase satisfies our intuition that hydrogen, being lighter, "must go up." This intuitive notion is given some mathematical underpinning if we define a gravitational scale height kT/mg, where kT is the thermal energy, m is the ion mass, and g is the acceleration of gravity. The density of the ion decreases with altitude as exp[−mgz/kT], where mgz is the gravitational potential energy. The proportion of H$^+$ increases with altitude because the scale height of H$^+$ is greater than that of other ions.
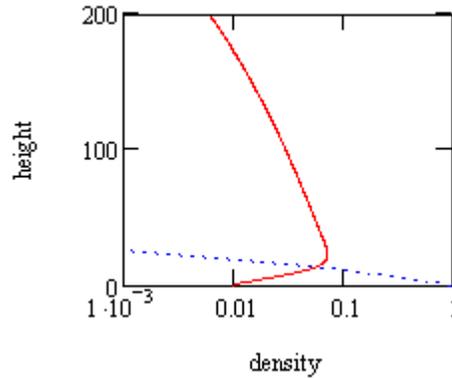
Density of Oxygen ions (dotted) and hydrogen ions (solid) as a function of height.

Now consider the electrons in the ionosphere. Their gravitational scale height is about 1836 times greater than that of H$^+$ and the electrons would be higher than the ions if there were no other effects. The electrons being higher, however, suggests an electric field pointing upward that will tend to draw the electrons and ions together. Quasineutrality is a strong requirement, thus we can conclude that the electric field equalizes densities. The total potential energy for each particle is the sum of the gravitational potential energy mgh and the electrostatic potential energy ±qΦ (depending on the sign of the charge). Setting the scale heights of electrons and H$^+$ ions equal, we obtain:

$$m_H gz + q\Phi = m_e gz - q\Phi \qquad \text{or} \qquad q\Phi = -\tfrac{1}{2}(m_H - m_e)gz$$

Differentiating Φ, we find the electric field that equalizes the net force:

$$E = \frac{g}{2q}(m_H - m_e)$$

where m_H is the mass of H$^+$ and m_e is the electron mass.

An ionosphere of electrons, H$^+$ and O$^+$

Suppose that there are electrons and two types of ions, H$^+$ and O$^+$. How do the densities of the three species vary with altitude? The ions are mostly O$^+$ at the bottom of the ionosphere and mostly H$^+$ at the top. Thus the mass of the ion in the expression above for the electric field is some varying "blend" of the two masses. There is no simple expression for E. E must be found by solving Poisson's equation. For the solution to this equation we will use the relaxation method reviewed on the next page. For the densities of the species, we will use the equilbrium values:

$$n_e(z) = n_{e0} \exp\left[-\frac{(m_e gz - q\Phi)}{T} \right]$$

$$n_H(z) = n_{H0} \exp\left[-\frac{(m_H gz + q\Phi)}{T} \right] \qquad n_O(z) = n_{O0} \exp\left[-\frac{(m_O gz + q\Phi)}{T} \right]$$

where T is the temperature in energy units (so that Boltzmann's constant does not appear).

Try it: Derive the expressions for the densities by noting that for species s the momentum equation for equilbrium is:

$$-\nabla P_s - n_s m_s g - n_s q_s \nabla \Phi = 0$$

For simplicity, we assume an isothermal ionosphere with:

$$\nabla P_s = T_s \nabla n_s$$

Relaxation method for Poisson's equation

The values of the potential Φ will be assigned on grid points equally spaced in z. For example, Φ_n is the value of $\Phi(z)$ at the location z_n, the nth grid point. The first equation below is Poisson's equation written in the usual way. The second equation is the definition of the z derivative of Φ_n, using values from the grid. The third equation defines the second derivative of Φ_n. In the last equation, Poisson's equation is written with the finite-difference form of $d^2\Phi/dz^2$. In the last equation, we use the present values of Φ_{n-1} and Φ_{n-1} to calculate the next iteration for Φ_n. The left side of the equation is the new guess based on the old guess that is on the right side. This is done repetitively to get better values for Φ_n. The iterations are stopped when the new guess is no longer significantly different from the old guess.

$$-\frac{d^2}{dz^2}\Phi(z) = \left(n_i(z) - n_e(z)\right)q/\varepsilon_0, \qquad n_i(z) = n_H(z) + n_O(z)$$

$$\frac{d\Phi}{dz} = \frac{1}{\Delta z}\left[\Phi_{n+1} - \Phi_n\right]$$

$$\frac{d^2\Phi}{dz^2} = \frac{1}{\Delta z^2}\left[\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}\right]$$

$$^{new}\Phi_n = \frac{1}{2}\left[^{old}\Phi_{n+1} + ^{old}\Phi_{n-1}\right] + \frac{1}{2}\left(\Delta z\right)^2\left[n_i(z_n) - n_e(z_n)\right]/\varepsilon_0$$

Equations in dimensionless units

In the second line below, the terms in Poisson's equation have been multiplied by constants so that the last two bracketed terms are dimensionless. We recognize that the first bracketed term is the Debye length squared and that this can be used to make the z derivatives dimensionless as well. The last three lines show the dimensionless versions of the variables.

$$-\frac{d^2}{dz^2}\Phi(z) = \left(n_i(z) - n_e(z)\right)q/\varepsilon_0$$

$$-\left(\frac{\varepsilon_0 T}{n_0 q^2}\right)\frac{d^2}{dz^2}\left(\frac{q\Phi(z)}{T}\right) = \left(\frac{n_i(z)}{n_0} - \frac{n_e(z)}{n_0}\right)$$

$$\frac{d^2}{d\tilde{z}^2}\tilde{\Phi}(\tilde{z}) = \tilde{n}_i(\tilde{z}) - \tilde{n}_e(\tilde{z})$$

$$\tilde{\Phi} = q\Phi/T, \qquad \tilde{n} = \frac{n}{n_0}, \qquad \tilde{z} = \frac{z}{\sqrt{\varepsilon_0 T/n_0 q^2}} = \frac{z}{\lambda_{Debye}}$$

In the pages below, we are using the dimensionless variables without the tildes on top.

The particle densities

The expressions for the particle densities must be written in the dimensionless units.

The paticle masses in SI units: $m_e := 9.11 \cdot 10^{-31}$ $m_H := 1.67 \cdot 10^{-27}$ $m_O := 16 \cdot m_H$

The dimensionless value of Φ was found by dividing $q\Phi$ by T. Similarly, we can find the dimensionless value of g by dividing mgz by T. However, this will make the atmosphere so many Debye lengths tall that too many grid points are required. In order to decrease the number of grid points, we will increase g so that the density falls by a factor of 100 in only 200 Debye lengths. This means 200 mg is assigned the value -ln(0.01). Then g is:

$$g := \frac{-\ln(0.01)}{200 \cdot m_H}$$ $g = 1.379 \times 10^{25}$ This large value of g makes $m_H gz$ near order unity.

The proportion of ions that are hydrogen will be made 1% at the lower boundary of the domain:

$n_{H0} := 0.01$ $n_{O0} := 0.99$ $n_{e0} := 1.00$

With the above value of g, the equilibrium densities in dimensionless units are:

$n_H(\phi, z) := n_{H0} \cdot \exp(-\phi - 1.0\, m_H \cdot g \cdot z)$ $n_e(\phi) := n_{e0} \cdot \exp(\phi)$ $n_O(\phi, z) := n_{O0} \cdot \exp(-\phi - m_O \cdot g \cdot z)$

The z grid

The grid spacing will be $\Delta z = 0.5$ Debye lengths. A vector (a matrix with one row) will contain the values of Φ. We will use 400 grid points so that the domain is 200 Debye lengths.
The variable Phi will be the electrostatic potential that we are finding.

$\Delta z := 0.5$ The grid spacing in Debye lengths

$jmax := 400$ $j := 0, 1 .. jmax$ There will be jmax+1 grid points.

$z_j := j \cdot \Delta z$ Define grid points.

The program loop

The program loop is similar to the one used earlier for the Debye length. The successive iterations for the values of the potential Φ_k are saved in a matrix Phi. The first line of the loop initializes the matrix Phi to zero. The first "for loop" puts the initial guess, PhiAnalytic, into the first row of Phi.

The initial guess to start the iteration process will be the analytic solution for Φ on page 1:

$PhiAnalytic_j := \dfrac{-m_H \cdot g}{2} \cdot z_j$ This is the analytic solution for Phi.
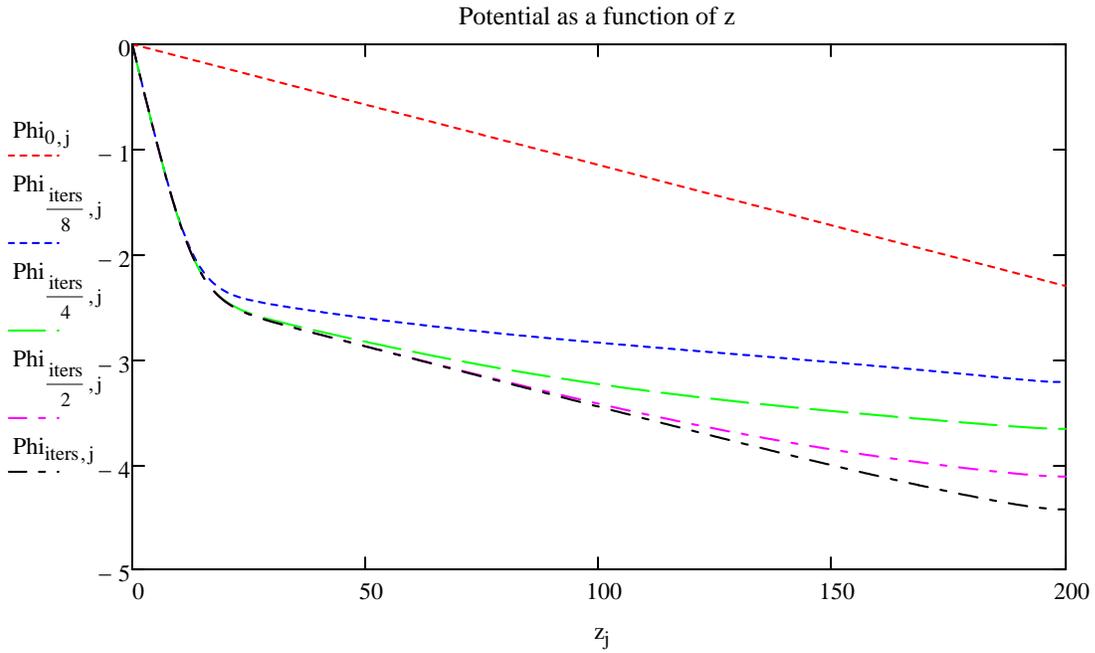
Boundary conditions

At the left boundary, the potential is specified to be zero. The "for j" loop in the program omits j = 0 so that the boundary value is preserved. We do not know the potential at the right boundary where j = jmax. A way to allow this value to relax toward an equilibrium is to use periodic boundary conditions. This is implemented by creating a fictitious point at jmax+1 that has the same value of potential as the point at jmax−1. The potential at jmax is figured by the usual formula, except that the potential value at the next (nonexistent) grid point jmax+1 is assigned the same value as the potential at the previous point. In the "for j" loop, the point at jmax is omitted and several lines are added after this loop that assign Phi at jmax using the rule for the periodic boundary.

Note that the new (temporary) potential value Temp is averaged with the previous potential value to prevent instability at the shortest wavelength.

$\text{iters} := 2048$ This is the number of iterations (found by trial and error)
needed to converge to a final solution for Phi.

$$\text{Phi} := \begin{vmatrix} \text{Phi}_{\text{iters},\text{jmax}} \leftarrow 0 \\[4pt] \text{for } j \in 0 .. \text{jmax} \\[2pt] \quad \text{Phi}_{0,j} \leftarrow \text{PhiAnalytic}_j \\[4pt] \text{for } i \in 1 .. \text{iters} \\[2pt] \quad \begin{vmatrix} \text{for } j \in 1 .. \text{jmax} - 1 \\[2pt] \quad \begin{vmatrix} \text{Temp} \leftarrow \dfrac{\text{Phi}_{i-1,j+1} + \text{Phi}_{i-1,j-1}}{2} \; ... \\[6pt] \qquad + \dfrac{1}{2} \cdot \Delta z^2 \cdot \left(n_H\!\left(\text{Phi}_{i-1,j}, z_j\right) + n_O\!\left(\text{Phi}_{i-1,j}, z_j\right) - n_e\!\left(\text{Phi}_{i-1,j}\right) \right) \\[6pt] \text{Phi}_{i,j} \leftarrow 0.5 \cdot \left(\text{Temp} + \text{Phi}_{i-1,j}\right) \end{vmatrix} \\[8pt] j \leftarrow \text{jmax} \\[4pt] \text{Temp} \leftarrow \dfrac{\text{Phi}_{i-1,j-1} + \text{Phi}_{i-1,j-1}}{2} + \dfrac{1}{2} \cdot \Delta z^2 \cdot \left(n_H\!\left(\text{Phi}_{i-1,j}, z_j\right) + n_O\!\left(\text{Phi}_{i-1,j}, z_j\right) - n_e\!\left(\text{Phi}_{i-1,j}\right) \right) \\[6pt] \text{Phi}_{i,\text{jmax}} \leftarrow 0.5 \cdot \left(\text{Temp} + \text{Phi}_{i-1,j}\right) \end{vmatrix} \\[6pt] \text{Phi} \end{vmatrix}$$

A plot of the successive values for Phi is on the next page.

**Potential as a function of z**

The potential initially decreases rapidly because of the smaller scale height of the O$^+$, then more slowly because of the H$^+$. The initial guess for the first iteration was a decrease with the scale height of H$^+$ (highest dashed line). At altitudes where O$^+$ is depleted, the scale height of H$^+$ applies.

Plot of the H$^+$ (solid line) and O$^+$ (dottend line) densities for the two-ion plasma:
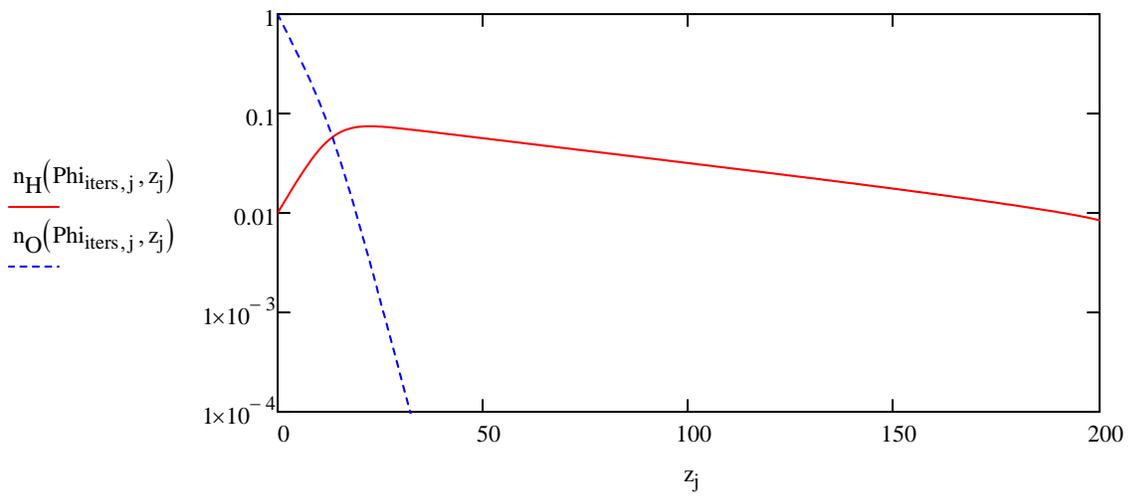
This is the same plot with altitude on the vertical axis:
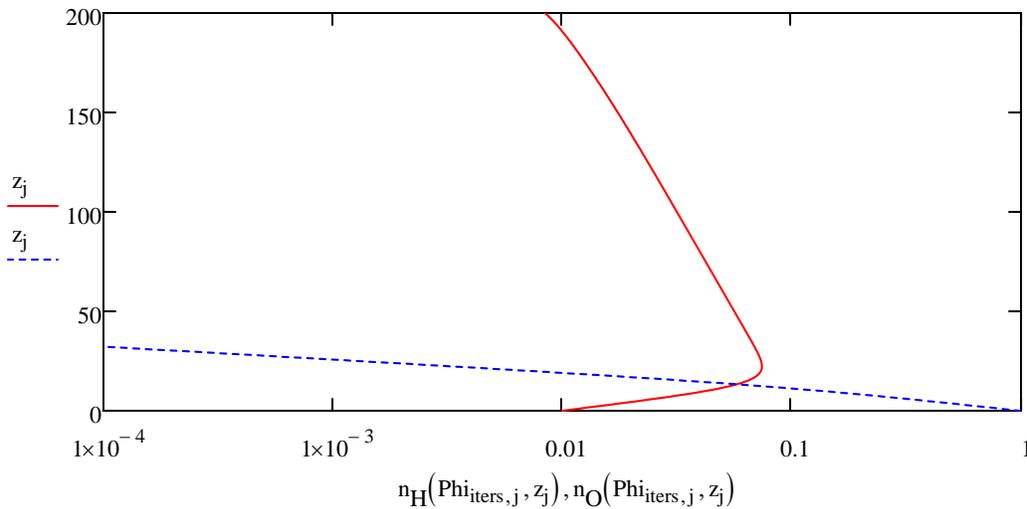
It seems odd that the H+ density increases at low altitudes. Is this really what happens? The answer is yes. In real units, the H+ density increases from 300 to 600 km, according to the plot in the reference below. The equilibrium value of E is sufficiently large at low altitudes to reverse the sign of the gradient in the H+. In the ionosphere, the equilibrium density is also affected by ionization and recombination which have not been considered here.

Try it: The periodic boundary conditions at jmax are equivalent to specifying E = 0 at the right boundary because these conditions impose mirror symmetry on Φ. Check that this periodic boundary condition has only a small effect on the solution by moving the boundary to 300 Debye lengths (jmax = 600). Observe that the converged solution at z = 200, $Phi_{iters,400}$, is not significantly changed if the boundary is move further to the right.

Try it: An alternate approach to finding the height dependence of the densities is to integrate the equilibrium momentum equations for the three species:

$$-T\nabla n_s - n_s m_s g + n_s q_s E = 0$$

The quasineutrality condition on the sum of the gradients provides a constraint that helps to find E in terms of the average ion mass. Find the expression for E and integrate the gradients in the three densities simultaneously using Runge-Kutta. Show that the result is the same as obtained using Poisson's equation.

Reference: Asgeir Brekke, *Physics of the Upper Polar Atmosphere*, Wiley-Praxis, Chichester, 1997, p. 230.