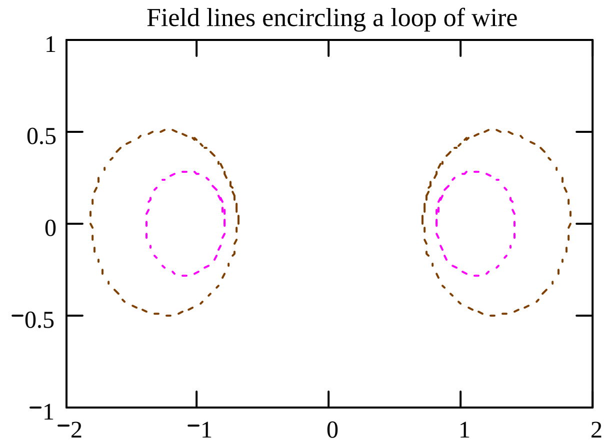
Field of a loop of wire

The expression for the magnetic field of a loop of wire is complicated. It involves the complete elliptic integrals of the first and second kind, K(k) and E(k), respectively. We can use the expression to make a plot for the B field lines around a current loop.



Complete elliptic integral of first kind

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-t^2)\cdot(1-k^2\cdot t^2)}}\,dt$$

of the second kind

$$E(k) := \int_0^1 \sqrt{\frac{1-k^2\cdot t^2}{1-t^2}}\,dt$$

These integrals will take a long time if we ask for a **tolerance** of 0.001. So we reduce it to 0.02

$TOL := 0.02$

The axis of our loop of wire will be in the z direction. We will plot in the x,z plane with y = 0.

Wire loop radius $a := 1$

For a loop of wire of radius a, the parameter k that is used above is defined as:

$$k(X,a) := \sqrt{\frac{4\cdot X_0\cdot a}{(X_0+a)^2+(X_2)^2}}$$

Our vector coordinate is X = (x,y,z). In our plots, x will be in the radial direction.

Our starting point is $X := \begin{pmatrix} 1.5 \\ 0 \\ 0 \end{pmatrix}$ x or r, y, z

Let's check if our functions are working

$k(X,a) = 0.98$ $K(k(X,a)) = 3.016$ $E(k(X,a)) = 1.051$

The r and z components of **B for a loop of wire** are given by these ugly expressions:

$$B_r(r,z) = \left[\frac{\mu I}{2\pi}\right]\frac{z}{r\sqrt{(r+a)^2+z^2}}\left[-K(k)+\frac{a^2+r^2+z^2}{(a-r)^2+z^2}E(k)\right]$$

$$B_z(r,z) = \left[\frac{\mu I}{2\pi}\right]\frac{1}{\sqrt{(r+a)^2+z^2}}\left[K(k)+\frac{a^2-r^2-z^2}{(a-r)^2+z^2}E(k)\right]$$

Let's avoid answers in scientific notation by setting $\mu/2\pi = 1$.
If you want answers in real (SI) units, replace I below with $\mu I / 2\pi$.

Also let the current to one. $I := 1$

In Mathcad notation, the fields are

$$Br(X) := \frac{I \cdot X_2}{X_0 \cdot \sqrt{\left(X_0 + a\right)^2 + \left(X_2\right)^2}} \cdot \left[-K(k(X,a)) + E(k(X,a)) \cdot \frac{\left[a^2 + \left(X_0\right)^2 + \left(X_2\right)^2\right]}{\left(a - X_0\right)^2 + \left(X_2\right)^2}\right]$$

$$Bz(X) := \frac{I}{\sqrt{\left(X_0 + a\right)^2 + \left(X_2\right)^2}} \cdot \left[K(k(X,a)) + E(k(X,a)) \cdot \frac{\left[a^2 - \left(X_0\right)^2 - \left(X_2\right)^2\right]}{\left(a - X_0\right)^2 + \left(X_2\right)^2}\right]$$

Let's check if these functions are working: $Bz(X) = -0.895$ $Br(X) = 0$

In the B field plotting exercise we learned that we could plot field lines by integrating a a unit vector with the length ds. That is what we will do here. The unit vector is in Derivs below.

The number of points evaluated will be $npoints := 20$

The starting point will be

$$X1 := \begin{pmatrix} .7 \\ 0 \\ 0 \end{pmatrix}$$

The unit vector projected onto x and z becomes

$$Derivs(s, X) := \begin{pmatrix} \dfrac{Br(X)}{\sqrt{Br(X)^2 + Bz(X)^2}} \\ 0 \\ \dfrac{Bz(X)}{\sqrt{Br(X)^2 + Bz(X)^2}} \end{pmatrix}$$

y is not used

The answers for this starting point will be in matrix M1.

$M1 := rkfixed(X1, 0, 4, npoints, Derivs)$

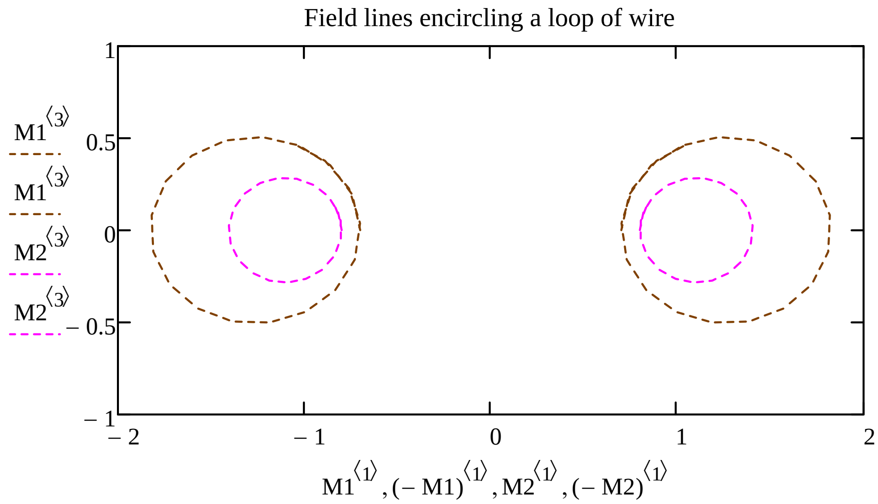
Let's also do a second starting point $npoints := 20$

$$X2 := \begin{pmatrix} .8 \\ 0 \\ 0 \end{pmatrix}$$

and put the second curve in M2

$M2 := rkfixed(X2, 0, 2, npoints, Derivs)$

The columns of the matrix are s, x, y, and z so we will plot column 3 versus column 1, which is z versus x. Also, because we know the problem is symmetric, we will also plot z against -x so we get the field lines that are on the left side also. The axis of the loop of wire is vertical and the loop itself passes through z = 0 at x = +1 and -1.



Field of a loop plus a constant field

Now let's add a vertically oriented field Bz0 to Bz. The field is now 3.14 on axis (see it at right), so let's add Bz = -4 so that the field is reversed on axis.

$$Bz\left(\begin{pmatrix}0\\0\\0\end{pmatrix}\right) = 3.142$$

Add it to the formula for Bz(X):

$$Bz0 := -4$$

$$Bz(X) := \frac{I}{\sqrt{\left(X_0 + a\right)^2 + \left(X_2\right)^2}} \cdot \left[K(k(X,a)) + E(k(X,a)) \cdot \frac{\left[a^2 - \left(X_0\right)^2 - \left(X_2\right)^2\right]}{\left(a - X_0\right)^2 + \left(X_2\right)^2}\right] + Bz0$$

Update the definition of the derivatives so it uses the new Bz

$$Derivs2(s,X) := \begin{pmatrix} \dfrac{Br(X)}{\sqrt{Br(X)^2 + Bz(X)^2}} \\ 0 \\ \dfrac{Bz(X)}{\sqrt{Br(X)^2 + Bz(X)^2}} \end{pmatrix}$$

More resolution: $npoints := 50$

The first plot:

$$X1 := \begin{pmatrix}.8\\0\\0\end{pmatrix}$$

$$M5 := rkfixed(X1, 0, 2, npoints, Derivs2)$$

The second plot:

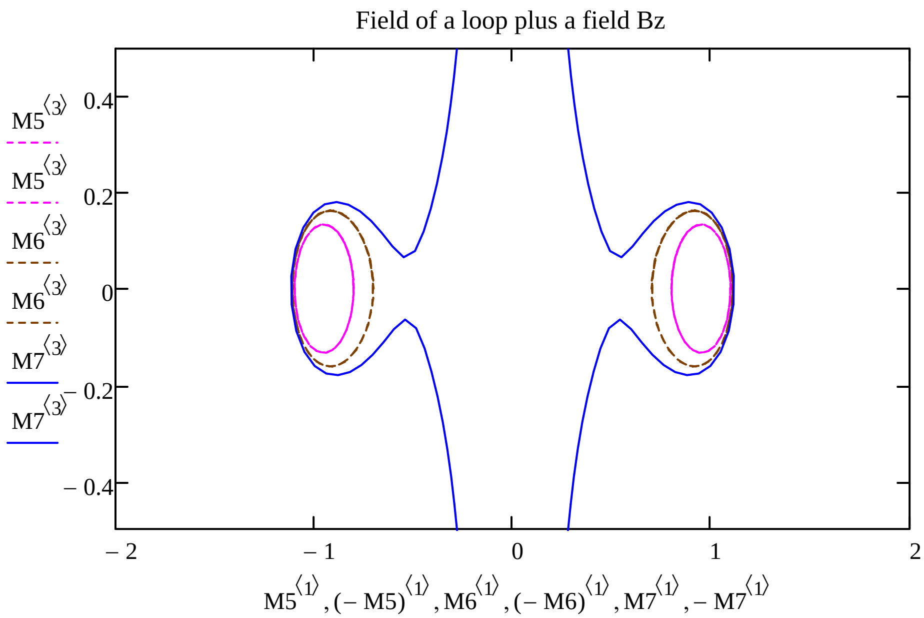
$$X2 := \begin{pmatrix}.7\\0\\0\end{pmatrix}$$

$$M6 := rkfixed(X2, 0, 3, npoints, Derivs2)$$

Let's do a third B field line starting at the top border of the graph (z = 0.5) with x = 0.28 :

$$X3 := \begin{pmatrix} .28 \\ 0 \\ 0.5 \end{pmatrix} \qquad M7 := \text{Rkadapt}(X3, 0, 3, \text{npoints}, \text{Derivs2})$$

Rkadapt will use smaller step sizes near the x point.



What happened? The field is "up" on the inside of the loop and down on the outside. We added "down" field everywhere. So the field lines are closer together now on the outside of the loop where the field was made larger in magnitude and are further apart on the inside of the loop where the magnitude is less. Of course, with our starting points being fixed at 0.7 and 0.8, these two lines must pass through those points so we don't see on the inside that the field lines are any further apart. We have forced them to be 0.1 apart. The field lines we plotted before (the magenta and brown lines) have moved.
The external field that we added contributes to the blue line. That downward pointing field is forced around the loop. There is an "x" point on both sides of the graph just a little outside of a radius of 0.5. The blue line hints at the "x" point. Our Runge-Kutta method will fail nearer to this point unless we use very tiny steps (npoints very large).

Try it: What happens if a line is started at X_0 = 0.25 so that it passes nearer to the x point? What happens if a line is started at z = 0.275? 0.27? etc. Can you "zero in" on the x point?

Try it: Can you make a 2-d vector plot of B like the plot on page 4 in B field of a wire? The difficult part will be defining the array of grid points so that a grid point doesn't fall on the loop, which would cause a division by zero.

Notes:
The field of a loop of wire is given in "Electromagnetic Theory" by Ernst Weber (Dover, New York, 1965) p.141. Note that the argument of the elliptic integrals k appears in the integrands as k^2. There are some math books that use m in place of k^2.