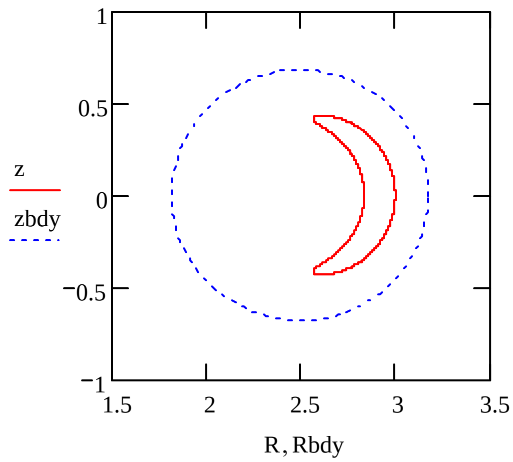
Banana orbits in Tokamaks

The toroidal magnetic field of the tokamak decreases with increasing major radius which causes a vertical drift of the charged particles. A single particles would drift out of the device if it did not also move parallel to the magnetic field lines. The poloidal field of the plasma current causes the field lines to spiral around the minor axis which prevents this loss. In lowest order, the motion of the guiding center follows the field line. In first order, if the gradient drift is upward the particles drift to larger minor radius when in the upper half of the torus and drift to smaller minor radius when in the lower half. The drift motion, when projected onto the minor radius, averages to zero and the particle is confined.



An ion orbit in the tokamak

If the particle has velocity parallel to the field line, the particle may be reflected by the mirror force as it moves to smaller major radius where the toroidal field is larger. In this case, the particle bounces back and forth between reflection points and the guiding center maps out a trajectory that is shaped like a banana when projected onto the plane perpendicular to the toroidal field (see the figure above).
In this exercise we will write down equations for the motion along the field line and add the vertical drift motion. The equations of motion will be integrated with the Runge-Kutta integrator to obtain a particle trajectory. This will be done for an impurity ion because the orbit is wider and easier to observe. SI units will be used and parameters will be taken from the TFTR tokamak that operated at Princeton Plasma Physics Laboratory in the 1980s.

Tokamak parameters:

These parameters are for an ohmically heated discharge in TFTR:

Major radius:	$R_0 := 2.5$	meters	Values at the minor axis will have subscript 0.
Minor radius:	$a := 0.68$	meters	
Safety factor:	$q := 2$	The safety factor is defined on the next page.	
Toroidal B field:	$B_{\phi 0} := 2.0$	Teslas	This is the value at the minor axis.

Ion parameters:

Ion energy	$W := 1000$	eV	This is a typical thermal energy for an ion.
Ion mass:	$m_i := 16 \cdot \left(1.67 \cdot 10^{-27}\right)$	kg.	This is the mass of an oxygen impurity ion.
Ion charge:	$q_i := 1.6 \cdot 10^{-19}$		Coulombs (not to be confused with the safety factor q.) The ion is assumed to be singly charged.

Ion speed: $v := \sqrt{2 \cdot \frac{q_i}{m_i} \cdot W}$ $v = 1.094 \times 10^5$ m/s

Ion cyclotron frequency: $\Omega_0 := \frac{q_i \cdot B_{\phi 0}}{m_i}$ $\Omega_0 = 1.198 \times 10^7$ radians per second at R_0.

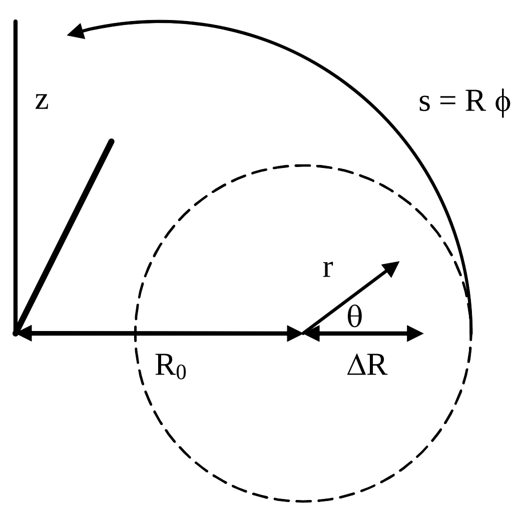
Larmor radius: $\rho := \frac{v}{\Omega_0}$ $\rho = 9.138 \times 10^{-3}$ meters

Motion along a field line:

We will use a cylindrical coordinate system (R, ϕ, z) and an auxiliary coordinate system (r,θ) for the plane perpendicular to the toroidal field that is centered on the minor axis. The *z* axis is the major axis of the torus. R_0 is the major radius of the torus and is a constant. The particle distance from the *z* axis is $R = R_0 + \Delta R$. An incremental distance measured along the minor axis ds = R dϕ, where ϕ is the toroidal angle. In the (r,θ) plane perpendicular to the minor axis, the horizontal axis is ΔR and the vertical axis is z. The poloidal field B_θ results in a field line spiraling around the minor axis. The field line moves through a small distance r dθ as it moves toroidally a distance R dϕ. The ratio of these distances is determined by the ratios of the poloidal field B_θ and toroidal field B_ϕ. From geometry: $r\, d\theta / R\, d\phi = B_\theta / B_\phi$.

The safety factor q has the following meaning. A particle following a field line will make q trips around the major axis when it has completed one spiral around the minor axis.

Hence, the definition: $$q = \frac{d\phi}{d\theta} = \frac{rB_\phi}{RB_\theta}$$



q is called the safety factor because operation of a tokamak below q = 1 on the axis results in instability. The safety factor is often slightly above unity near the axis and increases to about 3 at the wall.

The θ equation of motion for a particle following a field line is simply:

$$\frac{d\theta}{dt} = \frac{1}{q}\frac{d\phi}{dt}$$

Define a velocity vparallel that is the particle velocity parallel to the magnetic field line. The toroidal field lines are assumed nearly parallel to the minor axis, hence the parallel velocity is very near to the toroidal velocity:

$$vparallel = R\frac{d\phi}{dt} \quad \text{or} \quad \frac{d\phi}{dt} = \frac{vparallel}{R}$$

The distance along the field line is not precisely along any of the directions that we defined. This distance we will call *s* and its derivative is approximately:

$$vparallel = \frac{ds}{dt}$$

The velocity perpendicular to the field line will be called vperp.

The relationship between the Cartesian coordinates z, ΔR and the cylindrical coordinates r,θ are:

$$z = r\sin\theta \qquad \frac{dz}{dt} = r\cos\theta\left(\frac{d\theta}{dt}\right) = \Delta R\left(\frac{d\theta}{dt}\right) = \frac{\Delta R}{q}\left(\frac{d\phi}{dt}\right) = \left(\frac{\Delta R}{q}\right)\frac{vparallel}{R}$$

$$\Delta R = r\cos\theta \qquad \frac{dR}{dt} = -r\sin\theta\left(\frac{d\theta}{dt}\right) = -z\left(\frac{d\theta}{dt}\right) = \frac{-z}{q}\left(\frac{d\phi}{dt}\right) = -\left(\frac{z}{q}\right)\frac{vparallel}{R}$$

In defining the time derivatives, a long chain of substitutions has been used so that the final results are a function of z and R or ΔR..

Inpsection of the equations above shows that the solution for constant vparallel is a circle in the R,z plane, since the equations are of the form

$$\frac{dz}{dt} = C\Delta R \quad \text{and} \quad \frac{d\Delta R}{dt} = -Cz$$

where C is a constant.

Effect of the mirror force on vparallel

The mirror force is dependent upon the magnetic moment μ defined by: $\mu = \frac{m_i vperp^2}{2B_\phi}$

The mirror force F_s is directed along a field line and has magnitude $-\mu$ dB/ds:

$$F_s = m_i \frac{d}{dt} vparallel = -\mu\frac{dB_\phi}{ds} \qquad \text{Thus} \qquad \frac{d}{dt}vparallel = -\left(\frac{\mu}{m_i}\right)\frac{dB_\phi}{ds}$$

The way in which B_ϕ varies with position is: $B_\phi(R) = \frac{R_0 B_{\phi,0}}{R}$

The derivative of B_ϕ with respect to s is:
where we have used the chain rule.

$$\frac{dB_\phi}{ds} = \frac{dB_\phi}{dR}\frac{dR}{ds} = \frac{-R_0 B_{\phi,0}}{R^2}\frac{dR}{ds}$$

How does R vary with s? Recall that ds = R dϕ, dϕ = q dθ, hence ds = R q dθ and dθ = ds/Rq.

Since $R = R_0 + r\cos\theta$ then $\frac{dR}{d\theta} = -r\sin\theta = -z$

and $\frac{dB_\phi}{ds} = \frac{-R_0 B_{\phi,0}}{R^2}\frac{dR}{Rqd\theta} = \frac{zR_0 B_{\phi,0}}{qR^3}$

The final expression for the rate of change of vparallel is:

$$\frac{d}{dt} vparallel = -\frac{\mu\, z R_0 B_{\phi,0}}{q m_i R^3}$$

Initial conditions:

Use Insert|Units from the taskbar to specify that your are inputting a pitch angle in degrees. The default unit is radians.

The pitch angle ψ will be made large to minimize vparallel: $\psi := 72\text{deg}$ $\psi = 1.257$ radians

The initial velocities: $vparallel_1 := v \cdot \cos(\psi)$ $vperp_1 := v \cdot \sin(\psi)$

In SI units: $vparallel_1 = 3.382 \times 10^4$ m/s $vperp_1 = 1.041 \times 10^5$ m/s

The vertical drift velocity is: $vdrift := \frac{vperp_1^{\,2}}{2 \cdot R_0 \cdot \Omega_0}$ $vdrift = 180.902$ m/s

Note that the drift velocity is a very small fraction of vperp and vparallel.

The starting location of the impurity ion will be R_1: $R_1 := R_0 + 0.5 \cdot a$

The starting $B_{\phi 1}$ value is then: $B_{\phi 1} := B_{\phi 0} \cdot \frac{R_0}{R_1}$

The starting μ value is: $\mu := \frac{0.5 \cdot m_i \cdot vperp_1^{\,2}}{B_{\phi 1}}$ $\mu = 8.22 \times 10^{-17}$

A 4-vector Z will be defined that contains the coordinates R, z, ϕ, vparallel.
The starting coordinates will be called Zstart:

$$Zstart := \begin{pmatrix} R_1 \\ 0 \\ 0 \\ vparallel_1 \end{pmatrix} \qquad Zstart = \begin{pmatrix} 2.84 \\ 0 \\ 0 \\ 3.382 \times 10^4 \end{pmatrix}$$

R position
z position
ϕ position
vparallel

Integration of the equations of motion

For a detailed plot, we will want to have about 40 points in a radius R_0. The time to go this distance is approximately: $dt := \frac{1}{40} \cdot \frac{R_0}{v}$

We will follow the particle for a distance corresponding to about 18 trips around the torus, so the ending time will be $tEnd := \frac{18 \cdot (2 \cdot \pi \cdot R_0)}{v}$

The number of time steps will be:
This must be an integer. $npoints := \text{ceil}\left(\frac{tEnd}{dt}\right)$ $npoints = 4524$

The function DZ, a 4-vector, contains the time derivatives of R, z, ϕ, and vparallel.

$$DZ(t,Z) := \begin{bmatrix} \frac{-Z_1}{q} \cdot \left(\frac{Z_3}{Z_0}\right) \\ \left(\frac{Z_0 - R_0}{q}\right) \cdot \left(\frac{Z_3}{Z_0}\right) + vdrift \\ \frac{Z_3}{Z_0} \\ \frac{-\mu \cdot R_0 \cdot B_{\phi 0} \cdot Z_1}{m_i \cdot q \cdot (Z_0)^3} \end{bmatrix}$$

$$\frac{d\Delta R}{dt} = \frac{-z}{q}\frac{vparallel}{R}$$

$$\frac{dz}{dt} = \frac{\Delta R}{q}\frac{vparallel}{R} + vdrift$$

$$\frac{d\phi}{dt} = \frac{vparallel}{R}$$

$$\frac{d}{dt} vparallel = \frac{z\mu R_0 B}{q m_i R^3}$$

Note that ΔR is found using: $\Delta R = R - R_0 = Z_0 - R_0$ Z_0 is the zeroth component of Z.

rkfixed is used to find the successive values of the vector Z, which are put into the answer matrix M:

$$M := \text{rkfixed}(Zstart, 0, tEnd, npoints, DZ)$$

This is a listing of M

M =

	t	R	z	ϕ	vparallel
	0	1	2	3	4
0	0	2.84	0	0	$3.382\cdot10^{4}$
1	$5.711\cdot10^{-7}$	2.84	$1.259\cdot10^{-3}$	$6.8\cdot10^{-3}$	$3.382\cdot10^{4}$
2	$1.142\cdot10^{-6}$	2.84	$2.519\cdot10^{-3}$	0.014	$3.382\cdot10^{4}$
3	$1.713\cdot10^{-6}$	2.84	$3.778\cdot10^{-3}$	0.02	$3.382\cdot10^{4}$
4	$2.284\cdot10^{-6}$	2.84	$5.037\cdot10^{-3}$	0.027	$3.382\cdot10^{4}$
5	$2.856\cdot10^{-6}$	2.84	$6.296\cdot10^{-3}$	0.034	$3.381\cdot10^{4}$
6	$3.427\cdot10^{-6}$	2.84	$7.556\cdot10^{-3}$	0.041	$3.381\cdot10^{4}$
7	$3.998\cdot10^{-6}$	2.84	$8.814\cdot10^{-3}$	0.048	$3.381\cdot10^{4}$
8	$4.569\cdot10^{-6}$	2.84	0.01	0.054	...

Plots will be easier to make and interpret if we transform our R,z,ϕ coordinates to Cartesian coordinates x,y,z:

$\phi := M^{\langle 3 \rangle}$ $x := \overrightarrow{\left(M^{\langle 1 \rangle} \cdot \cos(\phi)\right)}$

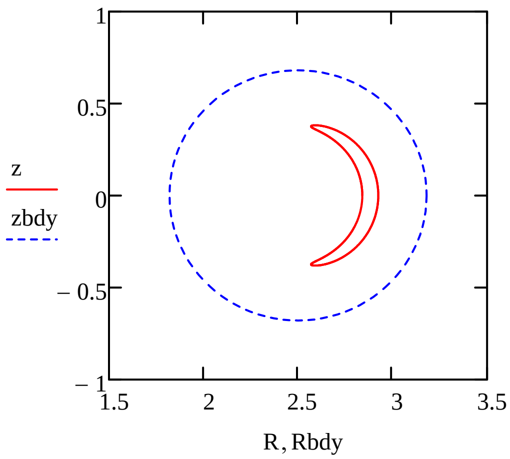
The arrow is the **vectorize** command from the matrix menu.

$z := M^{\langle 2 \rangle}$ $y := \overrightarrow{\left(M^{\langle 1 \rangle} \cdot \sin(\phi)\right)}$ $R := M^{\langle 1 \rangle}$

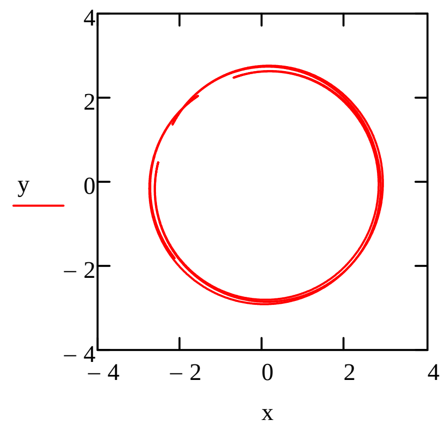
For plotting, it will be helpful to have the tokamak boundary defined: $kmax := 100$

$k := 0, 1 .. kmax$ $\theta_k := 2 \cdot \pi \cdot \frac{k}{kmax}$ $Rbdy_k := R_0 + a \cdot \cos(\theta_k)$ $zbdy_k := a \cdot \sin(\theta_k)$

This plot is a view of the banana orbit projected onto the R,z plane. Note the R values:



This is a plot looking from above:

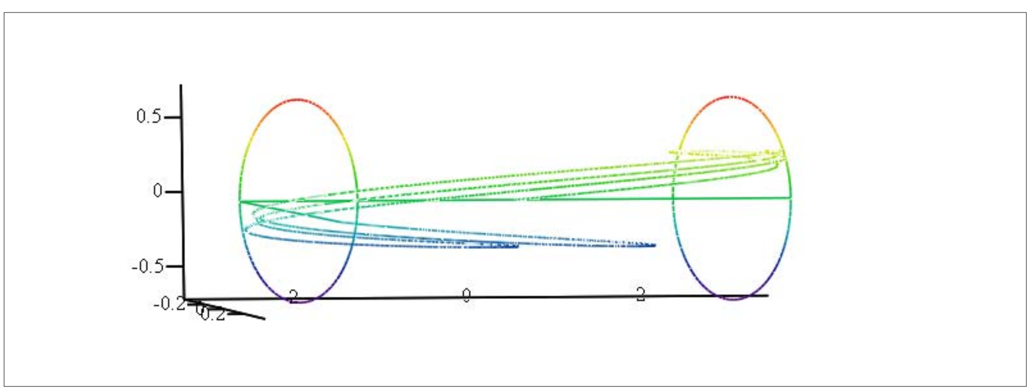


For the next plot, we will also put boundaries: $xbdy_k := 0$

$x := stack(z, xbdy, xbdy)$ $y := stack(y, -Rbdy, Rbdy)$ $z := stack(z, zbdy, zbdy)$

A point plot will be used and we have used the **stack** command to add these points to the plot of the banana. For clarity, the boundary is only displayed at two toroidal postions.

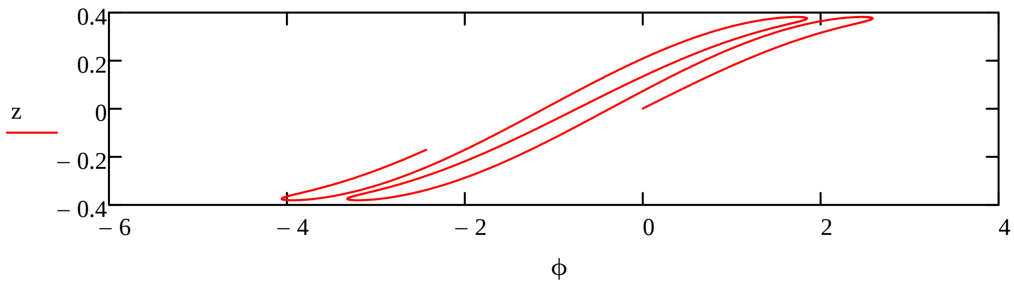
A 3-dimensional plot shows that the ion has mapped about two bananas:



(x, y, z)

Try it: Pull on a corner of the plot to see the banana orbit in three dimensions.
Colormap shading has been selected according to the z value. We see that the orbit is not closed in the x,y plane. The tips of the successive bananas are at smaller values of toroidal angle ϕ.

The failure of the banana orbits to close is easily seen in a z,ϕ plot:



Try it: Double the end time tEnd so that the bananas fill the torus. Tilt the axes of the 3-dimensional plot to see how the volume is filled.

Try it: Change the pitch angle ψ to a smaller value. Does the ion become a passing particle that circulates around the torus without being reflected by the mirror force?

Try it: Suppose the pitch angle is 90 degrees so there is no parallel motion. Does the ion vertically drift out of the torus?

Try it: How does the half-width of the banana orbit (seen in the z,R plot) compare with the poloidal Larmor radius $\rho_\theta = m_i$ vparallel / q_i B_θ? Recall that B_θ can be calculated using the definition of q.

Greater accuracy: allowing vdrift to change with position

Above it was assumed that the vertical drift velocity did not change with position. In fact, both vperp and vparallel change with R. Where the toroidal magnetic field is greater, vperp is also greater because the μ is conserved. The kinetic energy of the particle is conserved, hence vparallel must decrease if vperp increases. Below we will rewrite the expression for the drift velocity to include the dependence upon R.

vperp is easily found from the local B_ϕ using the magnetic moment:

If we calculate μ using the starting values at t = 0, the value of $vperp^2$ at later times is:

$$vperp^2 = \left(\frac{2\mu B_\phi}{m_i}\right)_{t=0} = vperp_1^2\left(\frac{B_\phi(R)}{B_{\phi,1}}\right) = vperp_1^2\left(\frac{R_1}{R}\right)$$

The relationship of the cyclotron frequency $\Omega(R)$ to the cyclotron frequency Ω_0 at the minor axis is: because of the gradient in B_ϕ.

$$\Omega(R) = \left(\frac{R_0}{R}\right)\Omega_0$$

The gradient and curvature drifts together are:

$$vdrift = \frac{0.5vperp^2 + vparallel^2}{R\Omega(R)} = \frac{0.5vperp^2 + vparallel^2}{R_0\Omega_0}$$

Conservation of kinetic energy:

The kinetic energy W is an invariant also: We will define W using the initial values of vperp and vparallel. The second equation will give us vparallel at subsequent times.

$$W = 0.5 m_i (verp_1^2 + vparallel_1^2)$$
$$vparallel^2 = 2W / m_i - vperp^2$$
$$vparallel^2 + 0.5 vperp^2 = 2W / m_i - 0.5 vperp^2$$

The definition of vdrift can then be simplified to: which depends only upon R and the initial conditions W, $vperp_1$ and R_1.

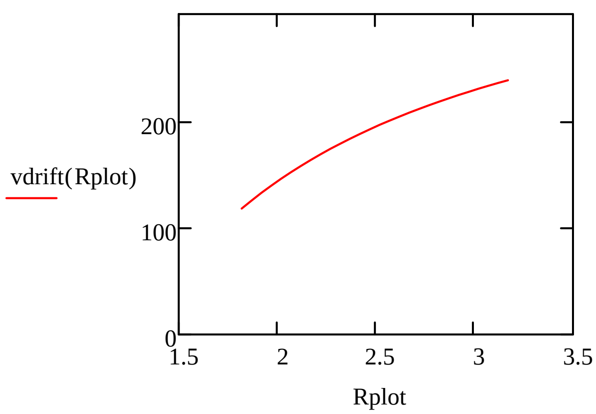
$$vdrift = \frac{2W / m_i - 0.5 vperp_1^2 \left(\frac{R_1}{R}\right)}{R_0 \Omega_0}$$

A new function for vdrift is then:

$$vdrift(R) := \frac{vparallel_1^{\,2} + vperp_1^{\,2} \cdot \left[1 - 0.5 \cdot \left(\frac{R_1}{R}\right)\right]}{R_0 \cdot \Omega_0}$$

Plot from the smallest R to the largest R:

$$Rplot := (R_0 - a), (R_0 - a) + 0.05 .. (R_0 + a)$$



This plot of vdrift(R) shows that the drift velocity changes significantly with major radius. At larger R, vperp is reduced by conservation of μ and vparallel is increaed from conservation of energy. The contribution of vparallel to the vertical drift is greater than the contribution of vperp.

Try it: Copy pages 5 and 6 and paste them below. Replace vdrift with vdrift(R). Compare the plots of the banana orbits with and without the R dependence.

Notes

1) It is an approximation that $(R+\Delta R)\, d\phi$ is the distance along a field line ds. More precisely, the projection of ds perpendicular to the x,R plane is $(B_\phi/|B|)ds$. This quantity is exactly $R d\phi$. The quantity |B| can be found from the vector components B_ϕ and B_θ. B_θ has not been specified but can be found from q, B_ϕ and the definition of q.

2) The vertical drift is in opposite directions for electrons and ions, thus the vertical drift results in a charge separation electric field (if the boundary is an insulator) and this field causes a drift of both species to larger major radius.

Reference

J. Wesson, *Tokamaks* (Clarendon Press, Oxford, 1997) 2nd edition, chapter 3.12.