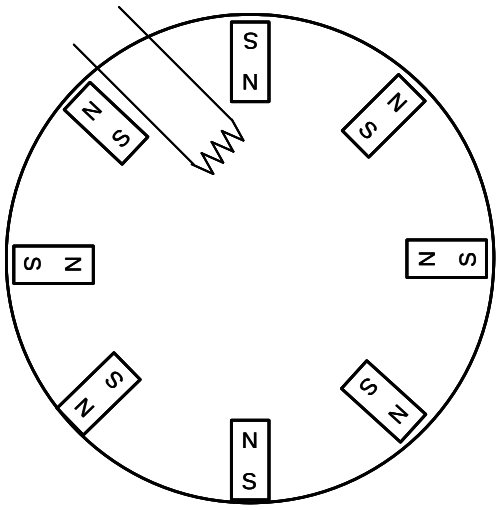
Particle orbits in the multidipole device



A simple way to make plasma is to place a heated emissive filament inside of a vacuum chamber with about 1 mTorr of gas. The filament is biased negatively and the electrons from the filament ionize the gas. The plasma density can be increased by lining the inside of the vacuum chamber with rows of magnets parallel to the cylinder axis. The rows of magnets should have alternating polarity as shown in the figure (see ref. 1). The filament is the zigzag line.

The magnetic field will be modeled by replacing each row of magnets with two wires in parallel, carrying opposite currents and separated by a distance comparable to the width of the magnet.

Assume a wire separation d, about the width of the magnets.
Half way between the two wires, the field of one of these wires is $\mu_0 I /2\pi(d/2) = \mu I /\pi d$.
Then for two wires, B half way between them is $2\mu_0 I /\pi d$.
Then $I = B\pi d /2\mu_0$ is the current in the wires to create a magnetic field equivalent to that of the magnets. A typical field at the magnet surface is 1000 Gauss or 0.1 Tesla.

$d := 0.01$ The wire separation is 1 cm.

$\mu_0 := 4\cdot\pi\cdot 10^{-7}$ A familiar constant.

$Bmax := 0.1$ The maximum field in Teslas.

$I := \frac{Bmax\cdot\pi\cdot d}{2\cdot\mu_0}$ This is the current in the wires to get the desired field.

$I = 1.25 \times 10^{3}$ amps This is the current to make the same field as the permanent magnets. This is a large current that would require water-cooled conductors. Permanent magnets are thus an advantage.

$r := 0.15$ Radius of cylindrical vacuum chamber in meters.

Polar coordinates will be used for the magnet locations. We need to decide the angular separation between the wires that mimic the magnets. The half angle of separation is half the separation d divided by r.

$\theta sep := \frac{d}{2\cdot r}$ $\theta sep = 0.033$ The half angle of the separation.

Find and plot the wire locations:

The wires are located around a circle with radius r. The angular locations of the magnet rows are multiples of $\pi/4$ radians. The wires in one pair have a spacing of $\pm$ 0.03 radians, approximately. With a radius of 0.15 m, this corresponds to 1 cm, about the width of the magnets.

$kmax := 8$ This is the number of rows of magnets.

Define the angular locations of the midpoints of the rows of wires which mimic the magnets:

$k := 0 .. kmax - 1$ $\theta_k := \left(\frac{k}{kmax}\right)\cdot 2\cdot\pi$

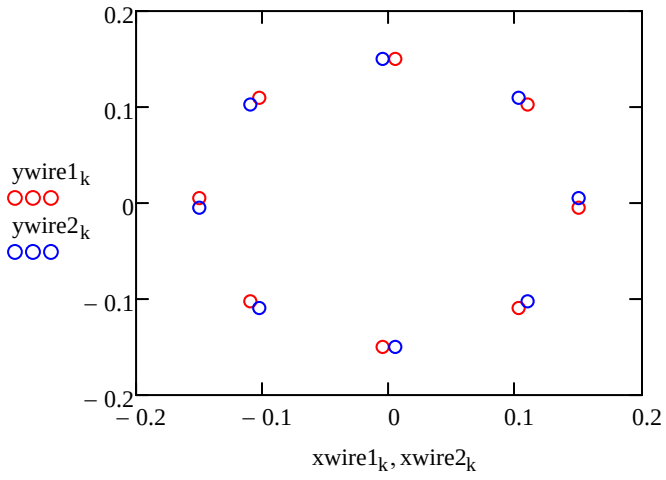
Convert r,θ locations to x,y locations: $xwire(\theta) := r\cdot\cos(\theta)$ $ywire(\theta) := r\cdot\sin(\theta)$

In a pair of wires, wire1 carries positive current and wire2 carries negative current.

$xwire1_k := xwire(\theta_k - \theta sep)$ $ywire1_k := ywire(\theta_k - \theta sep)$ Positive wire locations.

$xwire2_k := xwire(\theta_k + \theta sep)$ $ywire2_k := ywire(\theta_k + \theta sep)$ Negative wire locations.

Plot of the locations of the pairs of wires:



These stacks are the combined lists of the wire locations, for plotting later.

$xwires := stack(xwire1, xwire2)$

$ywires := stack(ywire1, ywire2)$

Calculate the B fields of the wire pairs:

We will plot the B vectors on an x,y grid with a grid spacing of 1 cm. The radius of the cylinder is 15 cm, thus we will go from -0.16 m to 0.16 m.

$imax := 32$ The maximum value of the counter for the number of grid points.

$i := 0, 1 .. imax$ the x counter $j := 0, 1 .. imax$ the y counter

$x_i := -0.16 + 0.32 \cdot \frac{i}{imax}$ $y_j := -0.16 + 0.32 \cdot \frac{j}{imax}$ These define the x,y grid from -0.16 m to +0.16 m.

Our x,y vector coordinate is $X := \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$ X_0 is x, X_1 is y

In terms of the components of X, the radial distance is: $r(X) := \sqrt{(X_0)^2 + (X_1)^2}$

The field of one wire:

$B\theta(X) := \text{if}\left[(r(X)) \neq 0, \frac{\mu_0 \cdot I}{2 \cdot \pi \cdot r(X)}, 10^{-4}\right]$ The if statement in this definition prevents an error if a divide by zero should occur.

These are the x and y components of B. Note that $B_x = (-y/r)B_\theta$ and $B_y = (x/r)B_\theta$.

$Bx(X) := \frac{(-X)_1 \cdot B\theta(X)}{r(X)}$ $By(X) := \frac{X_0 \cdot B\theta(X)}{r(X)}$

Sum the fields of the pairs of wires to get the multidipole field:

The two B_x values in parentheses below are the values for each of the wires in the pair.
We sum over kmax pairs of wires.

$$BX(x1, y1) := \sum_k \left[\cos(k \cdot \pi) \cdot \left[Bx\left(\begin{pmatrix} x1 - xwire1_k \\ y1 - ywire1_k \end{pmatrix}\right) - Bx\left(\begin{pmatrix} x1 - xwire2_k \\ y1 - ywire2_k \end{pmatrix}\right)\right]\right]$$

$$BY(x1, y1) := \sum_k \left[\cos(k \cdot \pi) \cdot \left[By\left(\begin{pmatrix} x1 - xwire1_k \\ y1 - ywire1_k \end{pmatrix}\right) - By\left(\begin{pmatrix} x1 - xwire2_k \\ y1 - ywire2_k \end{pmatrix}\right)\right]\right]$$

The cos ($k\pi$) alternates the sign of the rows of magnets.

Vector plot of the magnetic field:

The matrices MX and MY will contain the values of Bx and By at the grid points.

$MX_{i,j} := BX(x_i, y_j)$ $MY_{i,j} := BY(x_i, y_j)$

Arrow length: Mathcad scales the length of the arrows in a vector plot so that the longest one is one grid spacing. We will change the length of the arrow to the **fourth root** of the arrow length, which gives a nicer looking plot. (The short arrows are not too short to see.) If we do not do this, all the arrows will be about zero length except for the longest ones near the wire pairs. First, convert the vector components from x,y coordinates to polar coordinates. ABS1 will be the absolute value of the vector length and ANG will be the angle of the vector.

$ANG_{i,j} := angle(MX_{i,j}, MY_{i,j})$ $ABS1_{i,j} := \sqrt{(MX_{i,j})^2 + (MY_{i,j})^2}$

$MX2_{i,j} := \cos(ANG_{i,j}) \cdot \sqrt[4]{(ABS1_{i,j})}$ new x length of arrow.

$MY2_{i,j} := \sin(ANG_{i,j}) \cdot \sqrt[4]{(ABS1_{i,j})}$ new y length of arrow.

$RADIUS := 0.14$ Only plot vectors located inside the chamber radius. Set others to zero:

$MX2_{i,j} := if\left[(x_i)^2 + (y_j)^2 < RADIUS^2, MX2_{i,j}, 0\right]$

$MY2_{i,j} := if\left[(x_i)^2 + (y_j)^2 < RADIUS^2, MY2_{i,j}, 0\right]$

Question: What would be wrong with taking the fourth root of the components of the vectors MX and MY?

Also plot the magnetic vector potential:

An alternate way to visualize field lines is to plot the magnetic vector potential A. The magnetic field lines lie on surfaces of constant A. For a wire parallel to the z axis, the vector potential has only an axial z component A_z, and this varies logarithmically with distance.

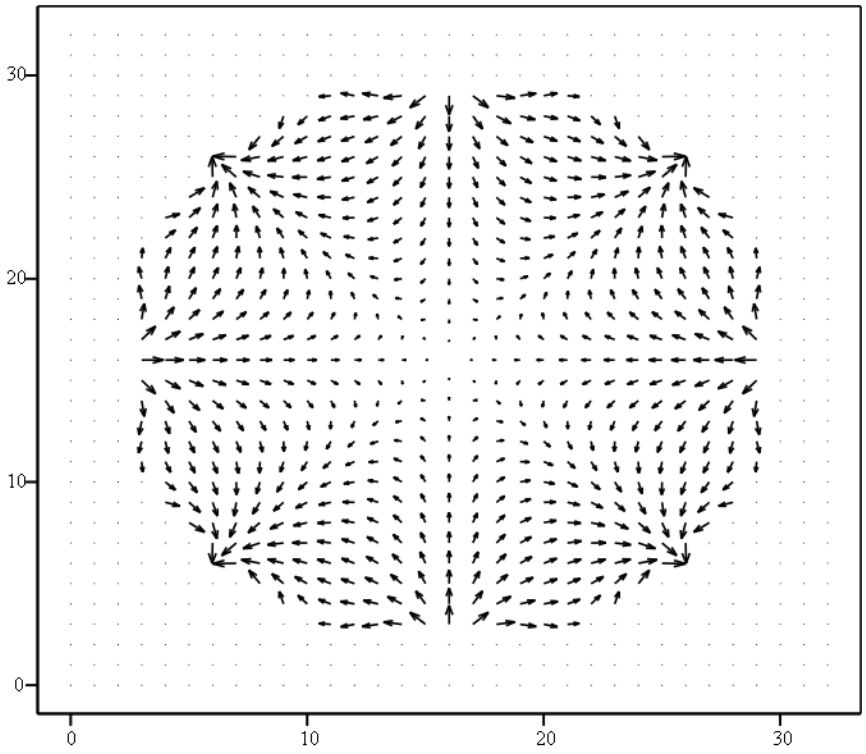
From $B_\theta = -dA_z/dr$, we find: $Az(X) := \frac{-\mu_0 \cdot I}{2 \cdot \pi} \ln(\sqrt{r(X)})$ The vector potential of one wire. The constant of integration is set to zero.

Z-component of the vector potential of the array of 8 pairs of wires:

$$AZ(x1, y1) := \sum_k \left[\cos(k \cdot \pi) \cdot \left[Az\left(\begin{pmatrix} x1 - xwire1_k \\ y1 - ywire1_k \end{pmatrix} \right) - Az\left(\begin{pmatrix} x1 - xwire2_k \\ y1 - ywire2_k \end{pmatrix} \right) \right] \right]$$

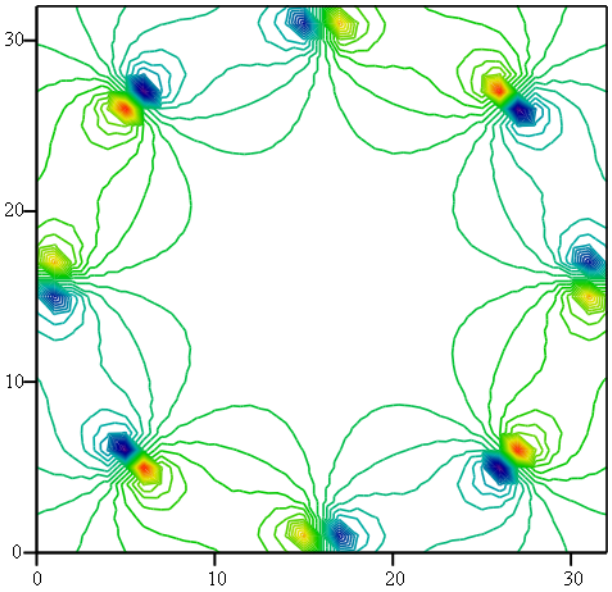
$MAZ_{i,j} := AZ(x_i, y_j)$ An array of vector potential values for plotting.

Vector B field plot for the multidipole device



(MX2,MY2)

Vector potential plot



MAZ

This plot of the vector potential contains nearly the same information as the B field vector plot. In this plot, the field outside the chamber is shown.

The vector potential A_z is zero on the midplanes and on the diagonals.

Use Runge-Kutta to find the electron motion:

Electron properties: $q := 1.6\cdot10^{-19}$ $m_e := 9.11\cdot10^{-31}$ $qmratio := \frac{q}{m_e}$

$W := 1\cdot q$ W is the electron kinetic energy, assumed to be 1 eV.

$v := \sqrt{\frac{2\cdot W}{m_e}}$ $v = 5.927\times10^{5}$ Electron velocity corresponding to 1 eV.

Pick starting X and V vectors for the electron:
We will stack the derivatives of x, y, and z above the derivatives of V_x, V_y and V_z and call the new 6-vector Z, below. The starting values at right were selected by trial and error to give nice looking plots.

$$X3 := \begin{pmatrix} 0.03 \\ 0.05 \\ 0 \end{pmatrix} \qquad V3 := \begin{pmatrix} 0.4\cdot v \\ 0 \\ v \end{pmatrix}$$

$Z := \text{stack}(X3, V3)$

Z is our 6-vector:

$$Z = \begin{pmatrix} 0.03 \\ 0.05 \\ 0 \\ 2.371\times10^{5} \\ 0 \\ 5.927\times10^{5} \end{pmatrix}$$

DZ is the derivative of the 6-vector Z and contains the Lorentz force:

$$DZ(t,Z) := \begin{bmatrix} Z_3 \\ Z_4 \\ Z_5 \\ -qmratio\cdot\left(Z_5\cdot BY\left(Z_0,Z_1\right)\right) \\ qmratio\cdot\left(Z_5\cdot BX\left(Z_0,Z_1\right)\right) \\ qmratio\cdot\left(Z_3\cdot BY\left(Z_0,Z_1\right) - Z_4\cdot BX\left(Z_0,Z_1\right)\right) \end{bmatrix}$$

dx/dt
dy/dt
dz/dt
dVx/dt
dVy/dt
dVz/dt

Time step for the Runge Kutta:

The absolute value of the B field at the starting location is:

$Babs := \sqrt{BY\left(Z_0,Z_1\right)^2 + BX\left(Z_0,Z_1\right)^2}$ $Babs = 2.081\times10^{-4}$

The gyro frequency at the starting location: $\Omega := qmratio\cdot Babs$ $\Omega = 3.655\times10^{7}$

We will integrate for 10 gyro periods: $t := 10\left(\frac{2\cdot\pi}{\Omega}\right)$ $t = 1.719\times10^{-6}$ seconds

The number of iterations needed is about 4 per gyro period or 8 π per orbit.

$npoints := \text{ceil}(4\cdot\Omega\cdot t)$ $npoints = 252$

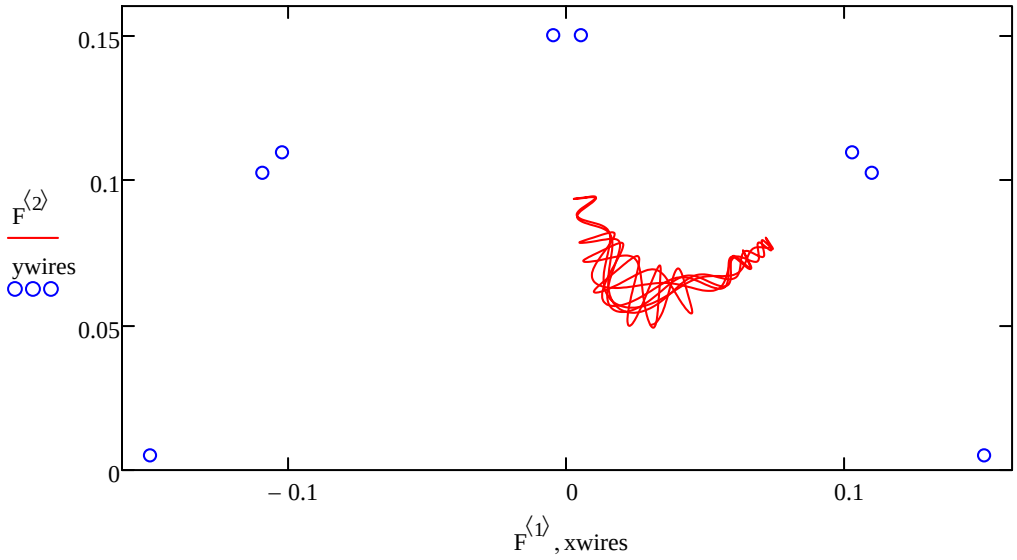
The B field is not uniform, so our time step should be shorter in regions of high field.
Thus to be conservative, we will greatly increase the number of points, which will decrease the time step:

$npoints := 4 \cdot npoints$

This is the approximate Larmor radius: $LarmorRadius := \frac{v}{(qmratio \cdot Babs)}$ $LarmorRadius = 0.016$

Now integrate to find the trajectory: $F := Rkadapt(Z, 0, t, npoints, DZ)$

This is a view down the axis of the device showing the wires and the mirroring of the electron.
Only the top half plane is plotted.

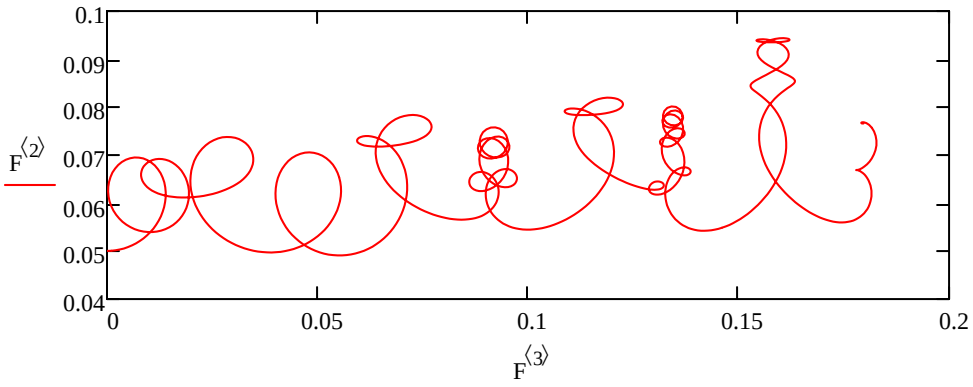


This is the anwer matrix F containing the electron positions and velocities:

F =

	t	x	y	z	Vx	Vy	Vz
0	0	0.03	0.05	0	$2.371 \cdot 10^{5}$	0	$5.927 \cdot 10^{5}$
1	$1.705 \cdot 10^{-9}$	0.03	0.05	$1.011 \cdot 10^{-3}$	$2.349 \cdot 10^{5}$	$3.71 \cdot 10^{4}$	$5.924 \cdot 10^{5}$
2	$3.411 \cdot 10^{-9}$	0.031	0.05	$2.019 \cdot 10^{-3}$	$2.322 \cdot 10^{5}$	$7.457 \cdot 10^{4}$	$5.899 \cdot 10^{5}$
3	$5.116 \cdot 10^{-9}$	0.031	0.05	$3.021 \cdot 10^{-3}$	$2.289 \cdot 10^{5}$	$1.124 \cdot 10^{5}$	$5.852 \cdot 10^{5}$
4	$6.821 \cdot 10^{-9}$	0.032	0.051	$4.014 \cdot 10^{-3}$	$2.251 \cdot 10^{5}$	$1.505 \cdot 10^{5}$	$5.781 \cdot 10^{5}$
5	$8.527 \cdot 10^{-9}$	0.032	0.051	$4.991 \cdot 10^{-3}$	$2.209 \cdot 10^{5}$	$1.888 \cdot 10^{5}$	$5.683 \cdot 10^{5}$
6	$1.023 \cdot 10^{-8}$	0.032	0.051	$5.95 \cdot 10^{-3}$	$2.165 \cdot 10^{5}$	$2.272 \cdot 10^{5}$	$5.558 \cdot 10^{5}$
7	$1.194 \cdot 10^{-8}$	0.033	0.052	$6.886 \cdot 10^{-3}$	$2.118 \cdot 10^{5}$	$2.657 \cdot 10^{5}$	...

The electron also drifts along z, as a consequence of the grad B drift. Below is a view looking along the x axis. The left and right sides of this plot are two ends of the cylindrical volume. The Runge-Kutta method does not conserve the magnetic moment very well, and part of the irregularity seen below is from the accumulation of error. The peaks in the y direction are from the electron approaching the pair of wires at the top of the previous figure.



Using interpolation to find B field values:

Above we calculated the values of B_x and B_y at every location of the electron. We can save time in our calculation by interpolating, using values previously calculated at the grid points. We will use the Mathcad cspline function which fits cubic polynomials to the B field values.

Put the x,y locations of the grid points into a two-column matrix: $Mxy := augment(x, y)$

$rows(MX) = 33$

Fit cubic splines separately to the x and y components of the B fields that were stored in matrices MX and MY:

$vx := cspline(Mxy, MX)$ $vy := cspline(Mxy, MY)$

Define functions bx and by that find the B field components between grid points by interpolation:

$$bx(x, y) := interp\left[vx, Mxy, MX, \begin{pmatrix} x \\ y \end{pmatrix}\right] \qquad by(x, y) := interp\left[vx, Mxy, MY, \begin{pmatrix} x \\ y \end{pmatrix}\right]$$

Use the interpolated fields in a new Runge-Kutta routine.

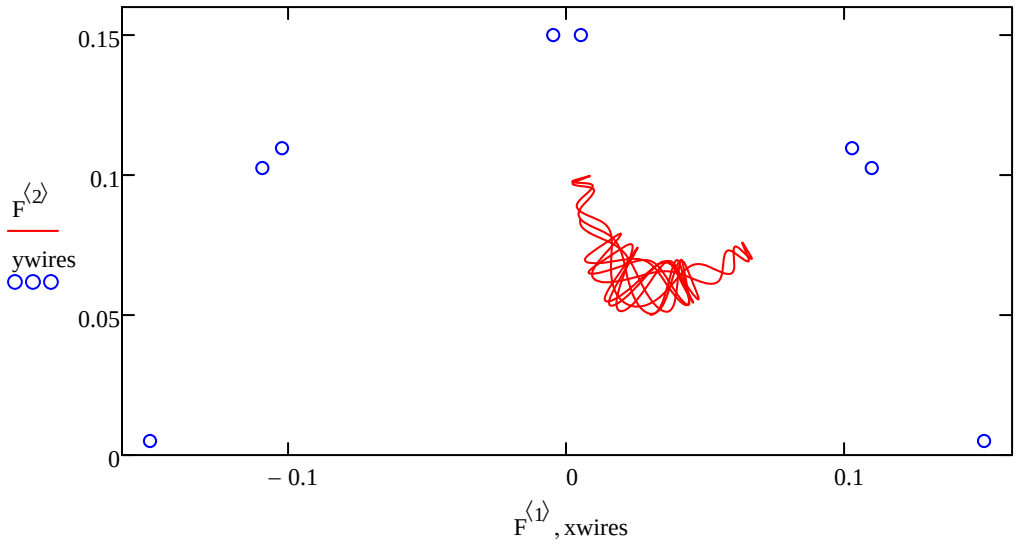
At right, the v_z B_y(x,y) component of the Lorentz force is written Z_5*by(Z_0,Z_1).

$$DZ(t, Z) := \begin{bmatrix} Z_3 \\ Z_4 \\ Z_5 \\ -qmratio \cdot \left(Z_5 \cdot by\left(Z_0, Z_1\right)\right) \\ qmratio \cdot \left(Z_5 \cdot bx\left(Z_0, Z_1\right)\right) \\ qmratio \cdot \left(Z_3 \cdot by\left(Z_0, Z_1\right) - Z_4 \cdot bx\left(Z_0, Z_1\right)\right) \end{bmatrix}$$

Find the trajectory using the interpolated fields: $F := \text{Rkadapt}(Z, 0, t, \text{npoints}, DZ)$

The trajectory calculated from interpolated B field values will be less accurate, but the calculation time will be reduced. .

Trajectory from interpolated values of B:



Chaotic orbit:

Suppose a particle begins near the center of the multidipole field. The field B value will be very small and the Larmor radius will be very large. The magnetic moment will not be conserved because the magnetic field changes in a length comparable to the Larmor radius.
This starting Z vector begins the particle near the origin.

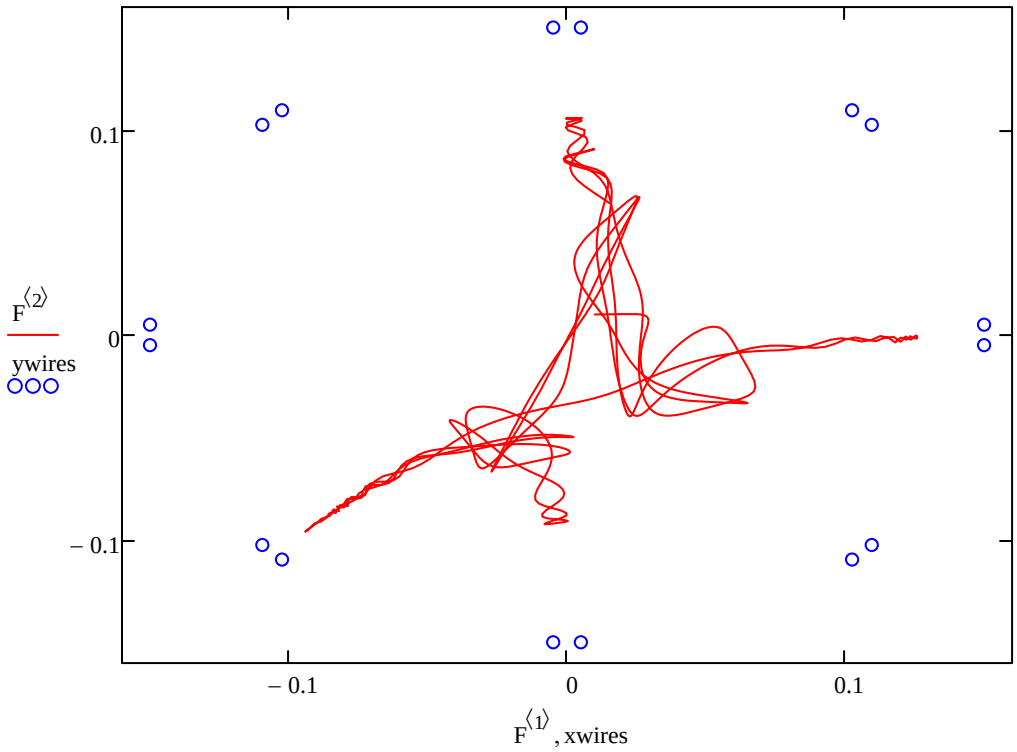
$$Z := \begin{pmatrix} 0.01 \\ 0.01 \\ 0 \\ 0.4 \cdot v \\ 0 \\ v \end{pmatrix}$$

We will integrate for 30 gyro periods:

$t := 30\left(\frac{2 \cdot \pi}{\Omega}\right)$ $t = 5.157 \times 10^{-6}$ seconds

Use Runge-Kutta to find the trajectory: $F := \text{Rkadapt}(Z, 0, t, \text{npoints}, DZ)$

Trajectory of an electron beginning near the axis:



About momentum conservation:
The canonical momentum $p_z = mv_z+qA_z$ is conserved because the fields are independent of z. Thus $v_z = qA_z/m$ + a constant. The absolute value of the axial velocity v_z is constrained. It cannot grow larger than the value $v_z = SQRT(2W/m_e)$ determined by the initial kinetic energy W of the electron (recall that B does no work). Thus the electron will never be found in regions where A_z is much different from the starting value. The most forbidden regions are near the wall half way between the wire pairs. If the electron is lost to the wall, it will be in a region of the wall where A_z is near zero. This occurs on the axes of symmetry which pass down the middle of the pairs of wires. If the electron is created on a surface with value A_z, it can stray a distance in A_z given by $dA_z = qv_z/m$. The z component of v will vary from $+v_z$ to $-v_z$ as the electron orbits the field line. The distance in real space that it can stray is $dr = (qv_z/m) /(dA_z/dr) = qv_z/mB_\theta$. In other words, the electron remains within about one Larmor radius of the surface with the starting value of A_z. Note that p_z is conserved even though the magnetic moment is not conserved near the axis.

References for multidipole devices:
1. R. J. Taylor, K. R. MacKenzie, and H. Ikezi, Rev. Sci. Instrum. 43, 1675 (1972).
2. E. R. Ault and K. R. MacKenzie, Rev. Sci. Instrum. 44, 1697 (1973).