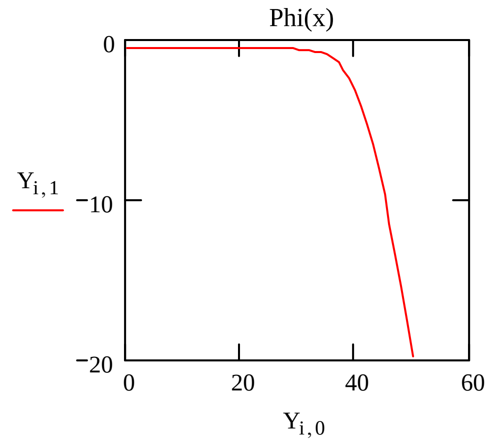
Bohm's sheath condition



Bohm wrote equations for the potential profile at the edge of the plasma (the sheath) and found that there was a minimum velocity for the ions. We wll reproduce his arguments using Poisson's equation with simple models for the electron and ion densities.

Poisson's equation: $-\nabla^2\Phi = (n_i - n_e)q/\varepsilon_0$

Bohm modeled the electron density using the Boltzmann relation: $n_e = n_0 \exp[q\Phi/T]$
where T is the temperature in energy units.

For the ions, he used the continuity equation: $\frac{d}{dx}(n_i v) = 0$

This equation can be integrated once to show that the ion current J = nqv is constant.
The ion velocity is obtained from conservation of energy. If the potential at infinity is assumed to the zero, and the local potential is Φ, then from conservation of energy the ion velocity v is the square root of -2qΦ/m. Φ is negative so the kinetic energy is positive. The ion density is then J/qv.

Switch to dimensionless units (see the exercise on Debye shielding) and define the dimensionless Φ as qΦ/T. The dimensionless ion velocity is obtained by dividing the ion velocity by the square root of T/m, where m is the ion mass.

Bohm found that he did not get a stable solution unless v > 1, in dimensionless units. This velocity requires that the starting potential at the sheath boundary be -0.5. This implies that the dimensionless electron density at the sheath edge is exp(-0.5) = 0.607. Quasineutrality at the sheath edge implies that this is also the ion density, so the dimensionless current at the sheath boundary is 0.607, the product of n and v at the boundary. The sheath is not quasineutral because the electron density falls to zero with distance more quickly than the ion density. The dimensionless sheath model equations are

$$\frac{d}{dx}\Phi = -E,$$
$$\frac{d}{dx}E = \frac{J}{v} - e^{-\Phi}$$

The starting values at the sheath boundary will be

$y := \begin{pmatrix} -0.5 \\ 0.001 \end{pmatrix}$ ϕ is yo
E is y1

We have used y_0 for Φ and y_1 for E.

The starting value for J is $J := e^{\frac{-1}{2}}$ $J = 0.607$

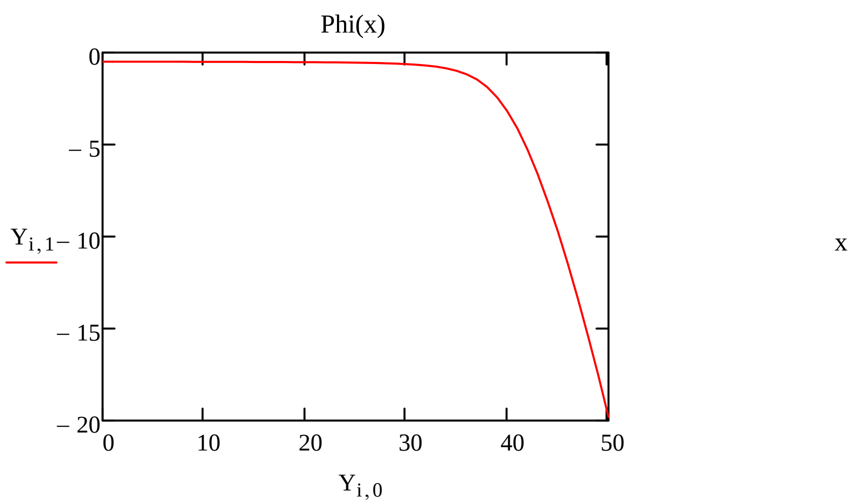
The differential equations are:

$$DY(x,y) := \begin{pmatrix} -y_1 \\ \frac{J}{\sqrt{-2 \cdot y_0}} - e^{y_0} \end{pmatrix}$$

The number of Runge-Kutta interations" $npoints := 50$ $i := 0, 1 .. npoints$

Start and end points $x1 := 0$ $x2 := 50$

The Runge-Kutta $Y := rkfixed(y, x1, x2, npoints, DY)$



Starting with E=0.001 causes the curve to dive downward between 40 and 50 Debye lengths. The region of steep descent is the sheath. If we make the starting E smaller, the distance to the start of the descent is increased. There is no single value for the thickness of Bohm's sheath, it depends on the starting value of E and the ending value for Φ. The sheath thickness goes to infinity as the starting E goes to zero.

The final value of the dimensionless potential Φ/T is: $Y_{npoints,1} = -19.809$

Try it: How does the sheath thickness change if the starting E is halved? doubled? Note that the scale on the Y axis changes and this can be confusing.

Shooting for the answer:
Suppose I know the potentials at the end points and don't know E?

Sometimes we are given second order differential equations and one boundary condition at each end. This is sufficient to determine an answer. However, to begin the integration we need two boundary conditions at the starting end. For Poisson's equation, we need to know both ϕ and E at the starting boundary. If instead, we are given ϕ at the left and right boundaries, we can find the potential profile by trial and error. We try different E values to start the integration and use the one that gives the potential specified at the other boundary. This procedure is called **shooting**. (If you "shoot" too high the first time, you aim lower.)

For example, the plasma center may be at potential 0 and the wall may be at -10 and located 20 Debye lengths away.

Then we must try different values for E at the left boundary, to see which curve passes through -10 at the right boundary.

$E := 0.01$ will be our first guess.

$$DY(x,y) := \begin{pmatrix} -y_1 \\ \dfrac{J}{\sqrt{-2\cdot y_0}} - e^{y_0} \end{pmatrix} \qquad y(E) := \begin{pmatrix} -.5 \\ E \end{pmatrix}$$

$x2 := 20 \qquad M(E) := \text{rkfixed}(y(E), x1, x2, \text{npoints}, DY)$

$M(E)_{\text{npoints},1} = -3.892$ With our initial guess of E = 0.0001, the final potential is not negative enough. Try a bigger E

M(E) is our matrix of answers and it is written as a function of the starting value of E, which we have named E. We want to find the value of E that makes the final value of the first column of the matrix equal to -10. Try a bigger E, 0.1 rather than 0.01:

$M(0.1)_{\text{npoints},1} = -22.621$ This is too negative. It will take a long time by trial and error to find the starting E that gives -10 for the final potential.

Let's use a root finder to find the value of E that makes the final value in the first column of the matrix, plus 10, equal to zero. First we make this a function:

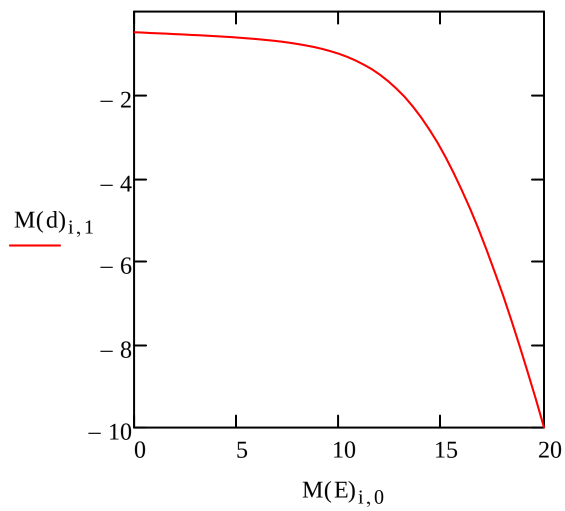
$f(E) := M(E)_{\text{npoints},1} + 10$ We have the right answer when this function zero.

Find it: $d := \text{root}(f(E), E)$

$d = 0.022$ This is the root that was found. We can test it

$M(d)_{\text{npoints},1} = -10$ Indeed this starting E generates a potential at the wall of -10.

We can check this answer by plotting it



The root finding function can find boundary conditions for you by trial and error!
Note carefully the procedure for what was done. The potential found by Runge Kutta at the right boundary was made "the answer" and it was expressed as a FUNCTION of the starting boundary condition. Then the root finder was used to find the argument of this function that gave the potential that we wanted at the boundary. This is a general procedure that can be used with other problems and with techniques than Runge Kutta.

Oscillatory solution and Bohm's velocity criterion

Bohm discovered that the equations have oscillatory solutions and do not give a monotonic sheath solution if the ions move more slowly. We can make them move more slowly by having them drop through a smaller potential. We will calculate their speed from the smaller potential (y_0 - 0.04):

$$y := \begin{pmatrix} -.5 \\ 0.001 \end{pmatrix}$$ ϕ is yo
E, is y1, starting values

The starting value for J above was $J = 0.607$

The differential equations must be written again with the new definition of velocity that uses y_0 - 0.04:

$$DY(x,y) := \begin{bmatrix} -y_1 \\ \dfrac{J}{\sqrt{-2\cdot(y_0 - 0.04)}} - e^{y_0} \end{bmatrix}$$

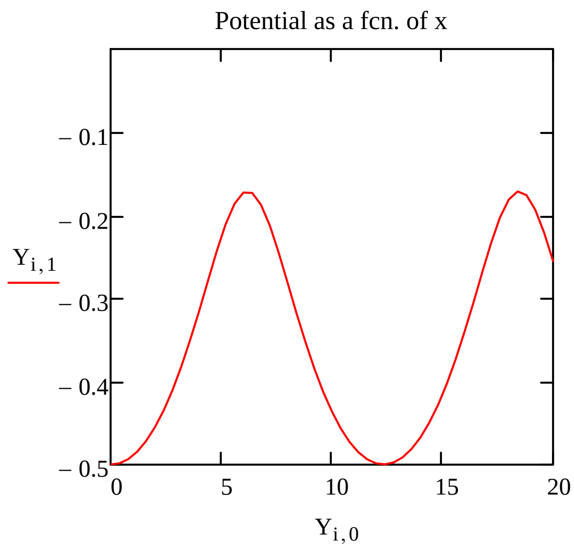
New starting and ending x:

$x1 := 0$

$x2 := 20$

The Runge-Kutta $Y := \text{rkfixed}(y, x1, x2, \text{npoints}, DY)$

Bohm 's sheath model does not give a monotonic sheath profile if the ions aren't moving at a dimensionless velocity of 0.5.



Indeed the solution is not monotonic, so this cannot be the observed sheath.

Reference:
Chapter 8 of Chen's textbook through 8.2.4.