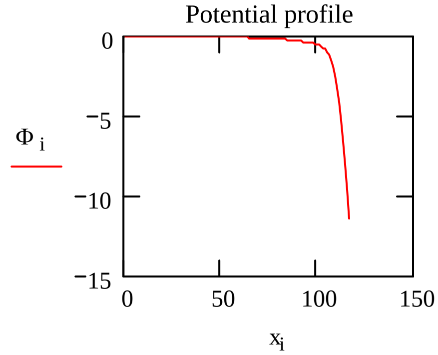
Planar sheath and presheath



A plasma between plane parallel walls develops a positive potential which equalizes the rate of loss of electrons and ions. The potential profile in the plasma can be found by integrating Poisson's equation from the plasma midplane to the wall. Models are needed for the electron and ion densities. It is customary to model the electron density using a Maxwellian distribution for which the density is given by a Boltzmann factor. The ions are modeled using fluid equations with a source term. The ion current to the wall is J and the charge density of ions is found by dividing J by the fluid velocity u.

1. Equations

$$\varepsilon_o \frac{d^2\Phi(x)}{dx^2} = -\frac{J(x)}{u(x)} + n_o q \exp\left\{\frac{e\Phi(x)}{T_e}\right\}$$ Poisson's equation

$$\frac{d}{dx}[nu] = R$$ Continuity equation with source R

It is assumed that the ions are created uniformly throughout the plasma volume with a rate R per unit volume per unit time. For our one dimensional plasma in a steady state, the current at a distance x from the midplane must be sufficient to take away the ions that are created between the midplane and x, which implies J = Rx. This result is obtained by integrating once the continuity equation. The momentum equation is:

$$\frac{d}{dt}nu + u\frac{d}{dx}nu + nu\frac{d}{dx}u - \frac{q}{m}E = 0$$ where E is the electric field.

Using the continuity equation, we can simplify the momentum equation:

$$nu\frac{du}{dx} = \frac{q}{m}nE - Ru$$

Note that the new ions are a sink for momentum because they are created with zero velocity and must be given momentum Ru per unit time in order to move at the fluid velocity. These differential equations can be solved simultaneously using Runge-Kutta or some other method.

It is useful to change to dimensionless variables:

$$\tilde{x} = x/\lambda_D \qquad \tilde{R} = R\lambda_D / n_0 c_s \qquad \tilde{\Phi} = q\Phi / T_e$$

$$\tilde{E} = eE\lambda_D / T_e \qquad \tilde{J} = J / n_o q c_s \qquad \tilde{u}_i = u / c_s$$

$$\tilde{J} = \tilde{R}\tilde{x} \qquad \tilde{n}_i = \tilde{J}/\tilde{u} \qquad \tilde{n}_e = e^{\tilde{\Phi}}$$ where $\lambda_D = \sqrt{\varepsilon_0 T_e / n_0 q^2}$

The equations for the problem, in dimensionless form are then:

$$\frac{d\tilde{E}(\tilde{x})}{d\tilde{x}} = \frac{\tilde{J}_i(\tilde{x})}{\tilde{u}_i(\tilde{x})} - \exp\left[\tilde{\Phi}(\tilde{x})\right] \qquad \frac{d\tilde{\Phi}(\tilde{x})}{d\tilde{x}} = -\tilde{E}(\tilde{x}) \qquad \frac{d\tilde{u}}{d\tilde{x}} = \frac{\tilde{E}}{\tilde{u}} - \frac{\tilde{R}}{\tilde{n}_i}$$

Through substitutions, the momentum equation can be rewritten: $\frac{d\tilde{u}}{d\tilde{x}} = \frac{\tilde{E}}{\tilde{u}} - \frac{\tilde{u}}{\tilde{x}}$

In the Mathcad version of the equation, the tildes are dropped for convenience.

2. Boundary conditions

We must begin the integration slightly away from x = 0 to avoid the division by zero that would occur in the momentum equation. u increases approximately linearly with x, thus the term u/x has a finite value near the origin.

There must be starting values for Φ, E, and u a short distance from the origin. Symmetry requires that Φ should be a function of even powers of x. A first approximation for Φ is then

$\tilde{\Phi}(\tilde{x}) = -\alpha\tilde{x}^2$ Thus $E(\tilde{x}) = 2\alpha\tilde{x}$ where α is a constant to be determined.

$\tilde{n}_e = 1$ is assumed at the midplane. From Poisson's equation we find that $\tilde{n}_i = (1 + 2\alpha)$

at the midplane. There is a slight excess of ions at the center because the ions are less mobile.

Using that $\tilde{J} = \tilde{R}\tilde{x} = \tilde{n}_i\tilde{u}$ we find for points near the origin $\tilde{u} = \frac{\tilde{R}\tilde{x}}{\tilde{n}_i} \approx \tilde{R}\tilde{x}$

This last relation provides us with a starting value for u at a small distance x_0 from the origin.

The following three relations give us the starting values for E, Φ, and u at the starting distance x_0:

$\tilde{\Phi}_0 = -\alpha\tilde{x}_0^2$ $\tilde{E}_0 = 2\alpha x_0$ $\tilde{u}_0 = \tilde{R}\tilde{x}_0$

3. A relation between R and α

A further look at the momentum equation, using values valid near the origin, shows that R and α cannot be chosen independently. The momentum equation can be solved for E to obtain

$\tilde{E}(\tilde{x}) = 2\left(\frac{\tilde{R}}{\tilde{n}_i}\right)^2 \tilde{x}$, but previously we showed $E(\tilde{x}) = 2\alpha\tilde{x}$

Thus it follows that $\tilde{R} = (1 + 2\alpha)\sqrt{\alpha}$ When R is sufficiently small: $\alpha \cong \tilde{R}^2$

4. A relation between the plasma size and R

A result of the sheath model is that the current at the wall is approximately 0.5 in dimensionless units. This means Rx = 0.5 at the wall. If the wall is a distance L from the midplane, RL = 0.5 and R = 0.5 / L, approximately. Thus the value of R to use can be found from the size of the plasma L.

5. Defining the problem

First we choose a size for the plasma of 100 Debye lengths: $L := 100$

Then $R := \frac{0.5}{L}$ $\quad R = 5 \times 10^{-3}$ $\quad \alpha := R^2$ $\quad \alpha = 2.5 \times 10^{-5}$

We must start the integration a short distance x_0 from the origin:
We use a distance 1% of the expected distance L. $x0 := 1$

We put the starting values of Φ, E and u into a vector y. Recall that the starting values are

$\tilde{\Phi}_0 = -\alpha \tilde{x}_0^2$ $\quad \tilde{E}_0 = 2\alpha x_0$ $\quad \tilde{u}_0 = \tilde{R}\tilde{x}_0$

The starting values will go in a vector y with the components Φ, E and u:

$$ystart := \begin{pmatrix} -\alpha \cdot x0^2 \\ 2\cdot\alpha\cdot x0 \\ R\cdot x0 \end{pmatrix}$$

ϕ is y0
E is y1
u is y2

$$ystart = \begin{pmatrix} -2.5 \times 10^{-5} \\ 5 \times 10^{-5} \\ 5 \times 10^{-3} \end{pmatrix}$$

ϕ is y0
E, is y1
u is y2

We will end the integration a little further than the estimated distance L:

$xend := ceil(L + 16)$

6. The derivatives for the Runge-Kutta integrator

$$DY(x,y) := \begin{pmatrix} -y_1 \\ \frac{R\cdot x}{y_2} - e^{y_0} \\ \frac{y_1}{y_2} - \frac{y_2}{x} \end{pmatrix}$$

The meaning of this is:

dϕ/dx= -E

dE/dx = n_i - n_e = J/u - exp(ϕ)

du/dx = E/u - u/x

$npoints := xend - 1$ the number of grid points. Note that the spacing is one Debye length.

$i := 0, 1 .. npoints$

7. Solution by adaptive Runge-Kutta:

$M := \mathrm{Rkadapt}(ystart, x0, xend, npoints, DY)$

For convenience, we assign the values in the answer matrix M to the variable names used above:

$x_i := M_{i,0}$

$\Phi_i := M_{i,1}$

$u_i := M_{i,3}$

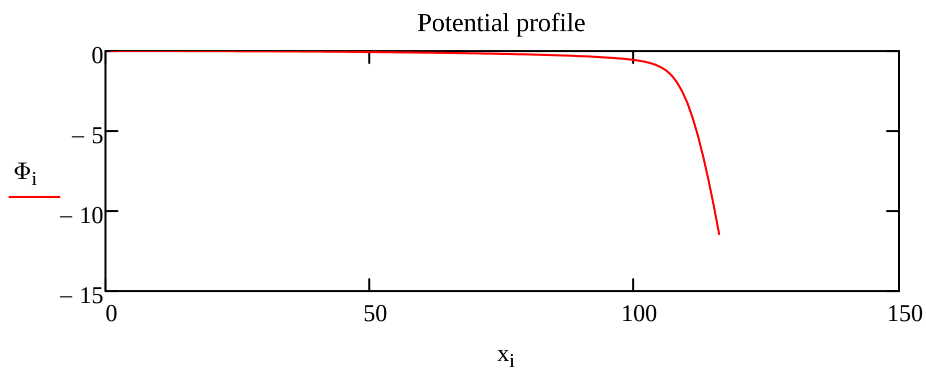
$n_{e_i} := e^{\Phi_i}$

$n_{i_i} := \dfrac{R \cdot x_i}{u_i}$

M =

	x (0)	φ(x) (1)	E(x) (2)	u(x) (3)
0	1	$-2.5\cdot10^{-5}$	$5\cdot10^{-5}$	$5\cdot10^{-3}$
1	2	$-1\cdot10^{-4}$	$9.7\cdot10^{-5}$	0.01
2	3	$-2.25\cdot10^{-4}$	$1.51\cdot10^{-4}$	0.02
3	4	$-4\cdot10^{-4}$	$1.99\cdot10^{-4}$	0.02
4	5	$-6.26\cdot10^{-4}$	$2.47\cdot10^{-4}$	0.03
5	6	$-9.01\cdot10^{-4}$	$3.16\cdot10^{-4}$	0.03
6	7	$-1.23\cdot10^{-3}$	$3.65\cdot10^{-4}$	0.04
7	8	$-1.6\cdot10^{-3}$	$4.19\cdot10^{-4}$	0.04
8	9	$-2.03\cdot10^{-3}$	$4.61\cdot10^{-4}$	...

Below is a plot of the potential profile in dimensionless units.
The vertical scale is qΦ/T and the horizontal scale is distance in Debye lengths.

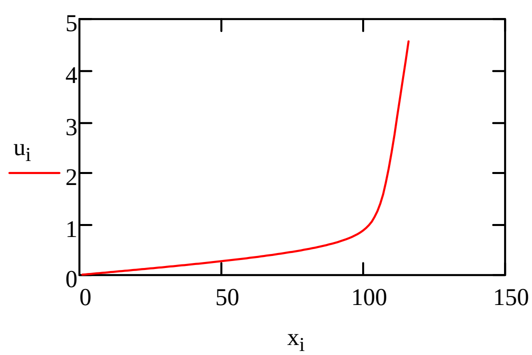


x

$\Phi_L = -0.618$

Our estimated distance L is *approximately* where Φ = -0.5.

Ion velocity u as a function of distance.

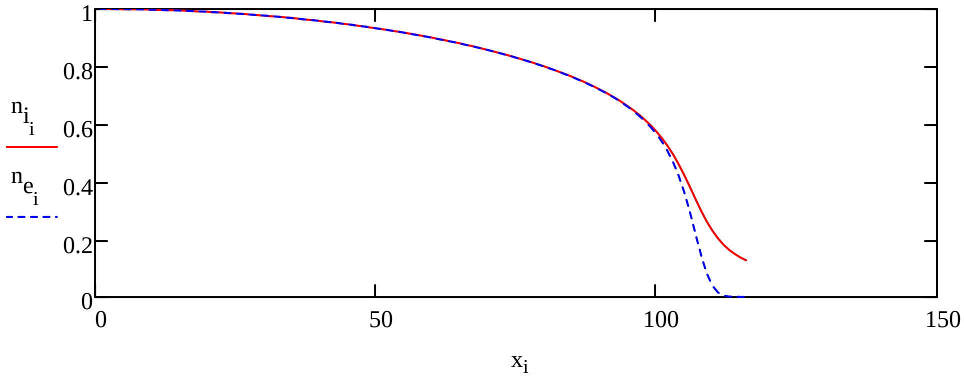


The ions become supersonic, u > 1, at *approximately* the boundary between the presheath and the sheath.

$u_L = 0.916$

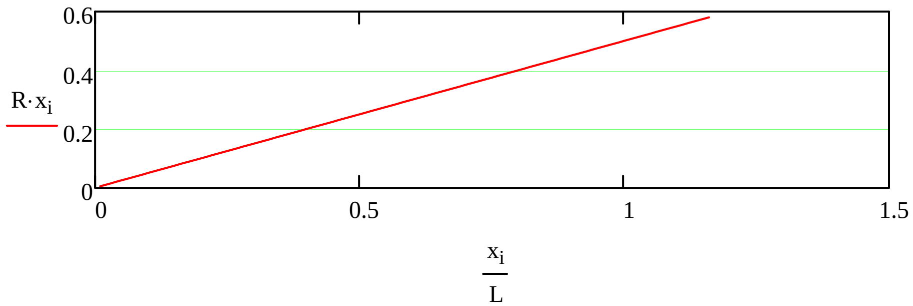
The boundary between the presheath and the sheath is somewhat arbitrary.

Electron and ion densities



Note that the electron density falls more rapidly as the wall is approached.
The plasma is approximately quasineutral, up to the distance for which Φ = -0.5.

Ion current to the wall



This trivial looking curve remains the same as L is changed. In dimensionless units, J is approximately 0.5 at the wall, always. The constancy of the curve shows that the ion current to the wall is nearly an invariant. The value of J = 0.5 n q c_s is often called the ion saturation current density.

Try it: Change L from 100 to 1000. It may be necessary to alter the defintion of xend by a few Debye lengths so that the graph of Φ descends to about -4, a typical value.

Notes:

1. It is difficult to find plots of this type in the literature.

2. There are problems with numerical stability if L is made larger than about 3000. Stability is improved by using the Bulirsch-Stoer integrator instead of the Runge-Kutta integrator. Simply substitute:

$$M := \text{Bulstoer}(ystart, x0, xend, npoints, DY)$$

3. There are discussions in the literature of solutions made by patching together solutions in the quasineutral region (made assuming $n_i = n_e$ exactly, sometimes called the "plasma approximation") and solutions made using Poisson's equation at the boundary. Modern computers can solve the full set of equations and there is no longer any motivation for patched solutions.

4. Experimentalists usually assign zero potential to the walls, thus the center of the plasma is at a positive potential. Theoretical work is often done with the center of the plasma taken as the zero of the potential scale. The zero point of the potential scale is arbitrary.

5. At what place should the integration be ended? The potential in the center of a plasma is typically 3 to 5 T_e/q greater than the wall potential. In models, a value for Φ is chosen so that the rate of loss of electrons and ions is equal. The simplest approximation is to find where the ion current J = Rx is equal to the random current of electrons, reduced by the factor $\exp[q\Phi/T_e]$. This simple assumption, however, is not likely to be valid because the tail of the electron distribution is not necessarily Maxwellian.

6. Can we have the integration end at Φ = -4, for example? There is no convenient way to have the integrator stop when a variable reaches a particular value. The loop below executes Runge-Kutta *one step at a time* and tests after each step whether or not Φ has reached the specified value. After each step, the new value for the variables is added to the end of the answer matrix N using the stack command and the values for x0 and ystart are updated so they can be used in the next iteration.

```
dx := 1        N := | M ← Rkadapt(ystart, x0, x0 + dx, 1, DY)
                    | N ← M
                    | ystart ← submatrix(M, 1, 1, 1, 3)^T
                    | x0 ← M[1,0]
                    | while M[1,1] > −4
                    |    | M ← Rkadapt(ystart, x0, x0 + dx, 1, DY)
                    |    | ystart ← submatrix(M, 1, 1, 1, 3)^T
                    |    | x0 ← M[1,0]
                    |    | N ← stack(N, submatrix(M, 1, 1, 0, 3))
                    | N
```

y0 should be a vector of one column, thus we use the matrix transpose operation to convert the last row of the answer matrix M to a column vector.

Reference:

Z. Sternovsky, Plasma Sources Sci. Technol. 14, 32-35 (2005).