Extraordinary Wave Dispersion Relation

The extraordinary wave is the linearly polarized electromagnetic wave propagating in a magnetized plasma with the electric vector and the wave vector perpendicular to B.
The dispersion relation is in Chen's plasma physics textbook, eqn. 4-104, and is

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{uh}^2}$$

where ω_{pe} is the electron plasma frequency and Ω_e is the electron cyclotron frequency.
The upper hybrid frequency is

$$\omega_{uh}^2 = \omega_{pe}^2 + \Omega_e^2$$

The roots of the dispersion relation can be found either of two ways: from the root finder function or the polyroots function.

I. Roots from the root finder function

First define some variables: $\omega_{pe} := 2$ $\Omega_e := 1.5$ $\omega_{uh} := \sqrt{\omega_{pe}^2 + \Omega_e^2}$

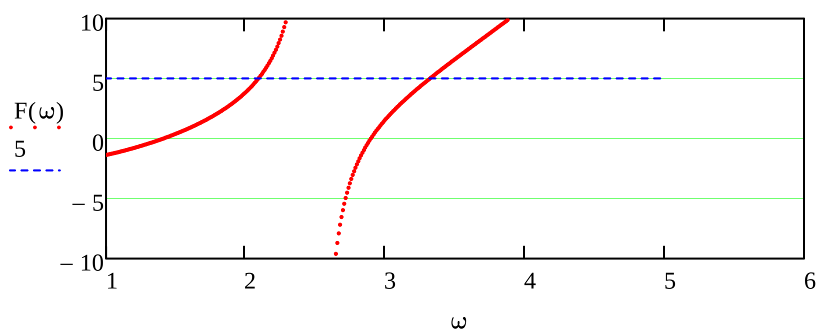
Multiply the dispersion relation above by ω^2, and the right hand side becomes

$$F(\omega) := \omega^2 - \omega_{pe}^2 \cdot \left(\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{uh}^2}\right)$$ which is also c^2k^2.

The dispersion relation is then $c^2k^2 = F(\omega)$. Ther plot below shows $F(\omega)$. Also is a trial value of 5 for c^2k^2. We see that the dispersion relation is satisfied for ω near 2 and near 3.

$\omega := 1, 1.01 .. 5$

The function $F(\omega)$



The locations where $F(\omega) = 5$ are the frequencies of the waves for which $c^2k^2 = 5$.

The root finder can find these roots. First we define a trial value for ω: $\omega\omega := 2$

$\text{root}(F(\omega\omega) - 5, \omega\omega) = 2.08$

And another trial value: $\omega\omega := 3$

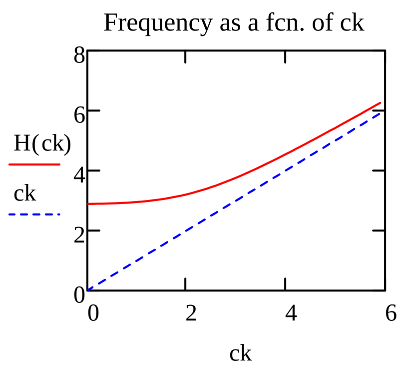
$\text{root}(F(\omega\omega) - 5, \omega\omega) = 3.305$

We can automate the procedure by making our answer a function of ck

$H(ck) := \text{root}\left[F(\omega\omega) - (ck)^2, \omega\omega\right]$

$ck := 0, 0.1 .. 5.9$

Now plot ω for a range of k:



This plot gives us the frequency for different values of ck.

For ck = 0, there are two roots called L (left) and R (right):

$$\omega_L := \frac{-\Omega_e}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_e^2}{4}}$$

$$\omega_R := \frac{\Omega_e}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_e^2}{4}}$$

$\omega_L = 1.386$

$\omega_R = 2.886$

The root finder returns: $H(0) = 2.886$

and does not find the other root. So for a complete answer we must use another method. The polyroots function finds all the roots of a polynomial expression, but to use it we must convert our dispersion relation into a polynomial.

II. Using polyroots to find both roots:

A little algebra on the dispersion relation yields

$$G(ck,\omega) := \omega^4 - \omega^2\cdot\omega_{uh}{}^2 - \omega^2\cdot\left(ck^2\right) - \omega^2\cdot\omega_{pe}{}^2 + \omega_{pe}{}^4 + \omega_{uh}{}^2\cdot(ck)^2$$

where G(ck, ω) = 0, so our answers are the roots of this function. Note that it is a biquadratic, meaning that it is quadratic in ω^2.

To use polyroots we must put the coefficients of the powers ω^2 of into a vector V. If we make this vector a function of ck, it will allow us to find roots for a range of values of ck.

$$V(ck) := \begin{bmatrix} \omega_{uh}{}^2\cdot(ck)^2 + \omega_{pe}{}^4 \\ -\omega_{uh}{}^2 - (ck)^2 - \omega_{pe}{}^2 \\ 1 \end{bmatrix}$$

The two roots for ck = 0 are

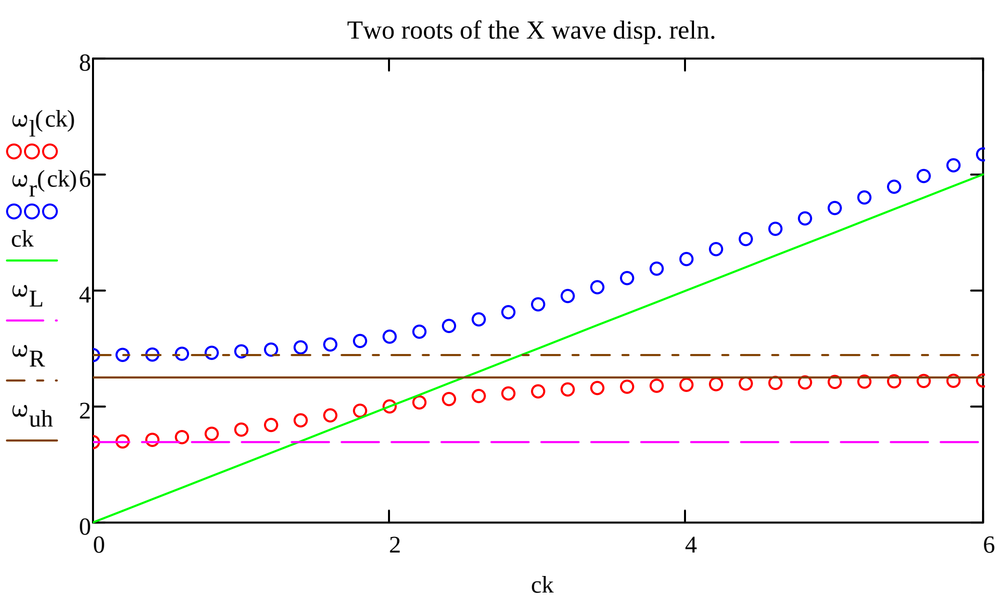
$$\text{polyroots}(V(0)) = \begin{pmatrix} 1.921 \\ 8.329 \end{pmatrix}$$

But these are the values of ω^2, not of ω.
The roots we are after are:

$$\sqrt{\text{polyroots}(V(0))_0} = 1.386 \qquad \sqrt{\text{polyroots}(V(0))_1} = 2.886$$

Let's make a range of values for ck: $ck := 0, 0.2 .. 6$

And define $\omega_l(ck) := \sqrt{\text{polyroots}(V(ck))_0}$ $\quad \omega_r(ck) := \sqrt{\text{polyroots}(V(ck))_1}$



Plotted on the graph are lines at the zeros ω_L and ω_R. The roots, plotted as circles, approach these values as ck goes to zero. For large ck, the asymptotic values are ck and ω_{uh}.

Above we had ω_{pe} = 2 and Ω_e = 1.5.
What does the above graph look like if the cyclotron frequency is larger than the plasma frequency?

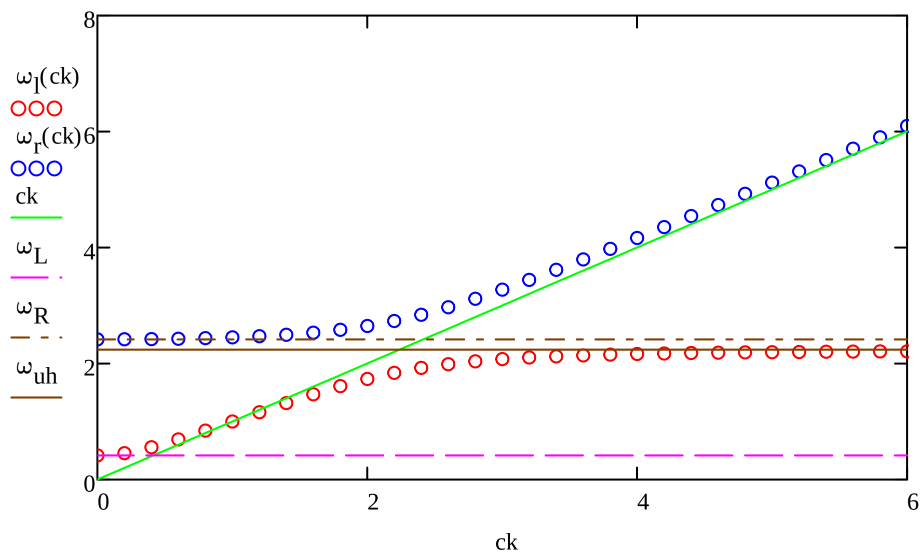
$\omega_{pe} := 1$ $\quad \Omega_e := 2$ $\quad \omega_{uh} := \sqrt{\omega_{pe}^2 + \Omega_e^2}$

$$V(ck) := \begin{bmatrix} \omega_{uh}^2 \cdot (ck)^2 + \omega_{pe}^4 \\ -\omega_{uh}^2 - (ck)^2 - \omega_{pe}^2 \\ 1 \end{bmatrix}$$

The subscripts l and r below select which of the two roots is being used:

$\omega_l(ck) := \sqrt{\text{polyroots}(V(ck))_0}$ $\quad \omega_r(ck) := \sqrt{\text{polyroots}(V(ck))_1}$

$$\omega_L := \frac{-\Omega_e}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_e^2}{4}} \qquad \omega_R := \frac{\Omega_e}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_e^2}{4}}$$



Try it:
Repeat this exercise for the left and right polarized waves propagating along B in section 4.16 of Chen's textbook.