Electrostatic Waves in Cold Magnetized Plasma

The dispersion relation for plane polarized waves in a cold magnetized plasma can be written

$$\omega^2 = \sum_s \frac{\omega_{ps}^2}{k^2}\left[k_z^2 + \frac{k_x^2}{1-\Omega_s^2/\omega^2}\right]$$

where the sum is over the electron and ion species (s=i or e).

We will begin by putting in some numbers.

$q := 1.602\cdot 10^{-19}\cdot \text{coul}$ $\varepsilon_0 := 8.854\cdot 10^{-12}\cdot\frac{\text{farad}}{\text{m}}$ $m_i := \frac{1.67\cdot 10^{-27}}{50}\cdot \text{kg}$ $m_e := 9.11\cdot 10^{-31}\cdot \text{kg}$

The ion mass above is 1/50 of the proton mass so that the cyclotron frequencies of electrons and ions are not so far apart. They will both appear on a graph away from the origin.

We will look at a laboratory plasma that is collisionless with

$n := 10^{16}\cdot \text{m}^{-3}$ $B := .004\cdot \text{tesla}$

Which gives

$\Omega_e := q\cdot\frac{B}{m_e}$ $\Omega_i := q\cdot\frac{B}{m_i}$ $\omega_{pe} := \sqrt{\frac{n\cdot q^2}{\varepsilon_0\cdot m_e}}$ $\omega_{pi} := \sqrt{\frac{n\cdot q^2}{\varepsilon_0\cdot m_i}}$

$\Omega_e = 7.034\times 10^8\,\frac{1}{s}$ $\Omega_i = 1.919\times 10^7\,\frac{1}{s}$ $\omega_{pe} = 5.641\times 10^9\,\frac{1}{s}$ $\omega_{pi} = 9.316\times 10^8\,\frac{1}{s}$

Note that we have $\omega_{pe} >> \Omega_e$. The graphs below will look different if the inequality is the other way.

We will choose an arbitrary wavevector k $k := 1\cdot \text{m}^{-1}$

And an angle of propagation relative to the magnetic field: $\theta := 0\cdot \text{rad}$

Which gives $k_z := k\cdot\cos(\theta)$ $k_x := k\cdot\sin(\theta)$ $k_x = 0\,\frac{1}{m}$ $k_z = 1\,\frac{1}{m}$

Divide the dispersion relation by ω^2 to get the new form: $1 - F(k_x,k_z,\omega) = 0$.
The dispersion relation is then satisfied where the curve $F(k_x,k_z,\omega)$ crosses 1.

$$F(k_x, k_z, \omega) := \frac{\omega_{pe}^2}{\omega^2 \cdot \left(k_x^2 + k_z^2\right)} \cdot \left(k_z^2 + \frac{k_x^2}{1 - \frac{\Omega_e^2}{\omega^2}} \right) + \frac{\omega_{pi}^2}{\omega^2 \cdot \left(k_x^2 + k_z^2\right)} \cdot \left(k_z^2 + \frac{k_x^2}{1 - \frac{\Omega_i^2}{\omega^2}} \right)$$

We expect to find roots at the upper and lower hybrid frequencies,

$$\Omega_{lh} := \sqrt{\frac{\omega_{pi}^2 + \Omega_i^2}{1 + \left(\frac{\omega_{pe}}{\Omega_e}\right)^2}} \qquad \Omega_{uh} := \sqrt{\omega_{pe}^2 + \Omega_e^2} \qquad \Omega_{lh} = 115.301 \cdot \text{MHz} \qquad \Omega_{uh} = 5.684 \times 10^3 \cdot \text{MHz}$$

at the plasma frequencies, $\omega_{pe} = 5.641 \times 10^3 \cdot \text{MHz}$ $\quad \omega_{pi} = 931.579 \cdot \text{MHz}$

$$\omega_{pie} := \sqrt{\omega_{pe}^2 + \omega_{pi}^2} \qquad \omega_{pie} = 5.717 \times 10^3 \cdot \text{MHz}$$

or at the frequency of electrostatic ion cycotron waves.

$$\Omega_{EIC} := \sqrt{\omega_{pi}^2 + \Omega_i^2} \qquad \Omega_{EIC} = 931.776 \cdot \text{MHz}$$

It is also possible to derive three approximate roots, $\Omega_s(\theta)$, $\Theta(\theta)$, and $\Xi(\theta)$, that apply for off-axis angles of propagation. These approximate roots are:

(letting $\theta = 0.01$ for now) $\quad \theta := 0.01 \cdot \text{rad}$

$$\Omega_s(\theta) := \Omega_{lh} \cdot \left[\frac{\omega_{pe}^2}{\Omega_e^2 + \omega_{pe}^2 \cdot (\sin(\theta))^2}\right] \cdot \left[1 + \frac{m_i}{m_e} \cdot (\cos(\theta))^2\right] \qquad \Omega_s(\theta) = 277.446 \cdot \text{GHz}$$

$$\Theta(\theta) := \frac{1}{\sqrt{1 + \left(\frac{\omega_{pe}}{\Omega_e}\right)^2 + \tan(\theta)^{-2}}} \cdot \sqrt{\omega_{pi}^2 + \left(\frac{\omega_{pe}}{\tan(\theta)}\right)^2} \qquad \Theta(\theta) = 5.622 \cdot \text{GHz}$$

For the static plasma dielectric constant we have: $\quad \varepsilon_p := 1 + \frac{\omega_{pi}^2}{\Omega_i^2} \qquad \varepsilon_p = 2.359 \times 10^3$

and from this we define our last θ-dependent frequency:

$$\Xi(\theta) := \sqrt{\frac{\omega_{pi}^2 + \omega_{pe}^2}{\varepsilon_p}} \cdot \cos(\theta) \qquad \Xi(\theta) = 1.177 \times 10^8 \, \frac{1}{s}$$

Case 1: Propagation parallel to B (θ = 0 degrees)

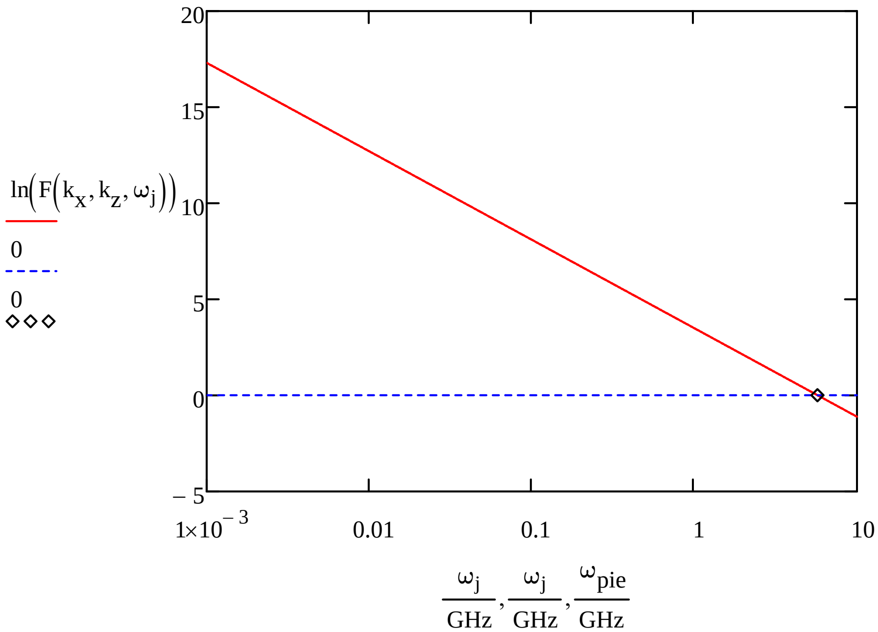
$k_x = 0\frac{1}{m}$ $\quad k_z = 1\frac{1}{m}$

We will use 800 values for ω logarithmically spaced from 10^6 to 10^{10}.

$j := 0..800 \qquad \omega_j := 10^{\frac{j+1200}{200}} \cdot \frac{1}{s}$

The graph will be made logarithmic on both axes because the values of F and ω vary over a wide range. The dispersion relation is satisfied when ln(F) = ln(1) = 0.

F(ω) for propagation parallel to B



Also plotted is a dotted line at zero (0 plotted vs. ω_j) and a black diamond at ω_{pie}.

Use the root finder to find exactly where F crosses 1:

$\mathrm{root}\left(F(k_x, k_z, \omega) - 1, \omega, 10^9 \cdot Hz, 10^{10} \cdot Hz\right) = 5.717 \times 10^9 \frac{1}{s}$ This is ω_{pie} which is a little larger than ω_{pe}.

For comparison $\qquad \omega_{pie} = 5.717 \times 10^9 \frac{1}{s}$

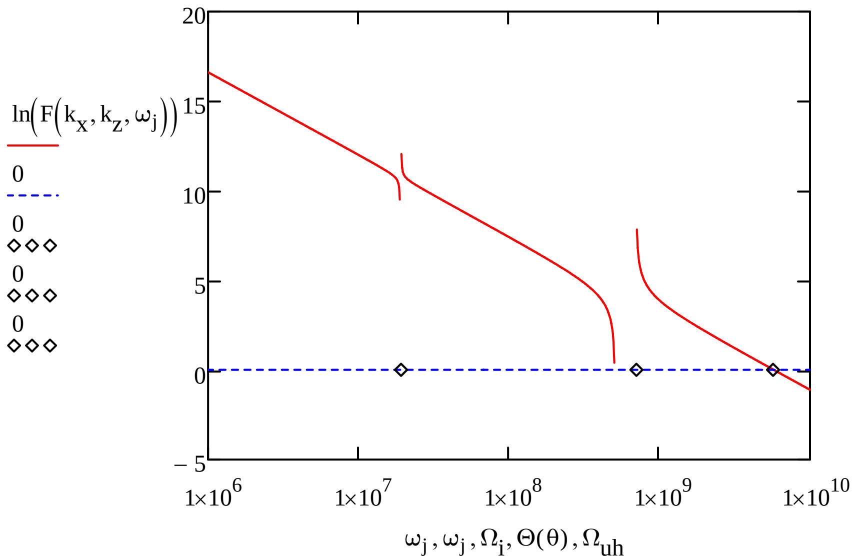
For propagation parallel to B, the only wave we find is at the combined plasma frequencies.

We used ω inside the root finder. Did doing this change the value of ω? The next graph uses the range of values for ω defined for the graph above, so using ω in the root finder did not alter our previous definition of ω as a range variable.

Case II. Propagation 45 degrees to B

$\theta := 45 \cdot deg$ $k_z := k \cdot \cos(\theta)$ $k_x := k \cdot \sin(\theta)$ $k_x = 0.707 \frac{1}{m}$ $k_z = 0.707 \frac{1}{m}$

F(ω) for 45 degree propagation



Use the root finder to find exactly where F last crosses 1, looking between 10^9 and 10^{10}:

$$\mathrm{root}\left(F(k_x, k_z, \omega) - 1, \omega, 10^9 \cdot Hz, 10^{10} \cdot Hz\right) = 5.738 \times 10^9 \frac{1}{s}$$

which is very nearly the upper hybrid frequency: $\Omega_{uh} = 5.684 \times 10^9 \frac{1}{s}$

And there is probably a root near Ω_i. Use that as a guess and see what happens

$A := 0.99 \cdot \Omega_i$ $\mathrm{root}\left(F(k_x, k_z, A) - 1, A\right) = 1.894 \times 10^7 \frac{1}{s}$ $\Omega_i = 1.919 \times 10^7 \frac{1}{s}$ Very close!

Last, there is a root near $B := 5 \cdot 10^8 \cdot \frac{1}{s}$ which we will use as a guess to obtain

$\mathrm{root}\left(F(k_x, k_z, B) - 1, B\right) = 5.021 \times 10^8 \frac{1}{s}$ which we can compare with the other frequencies.

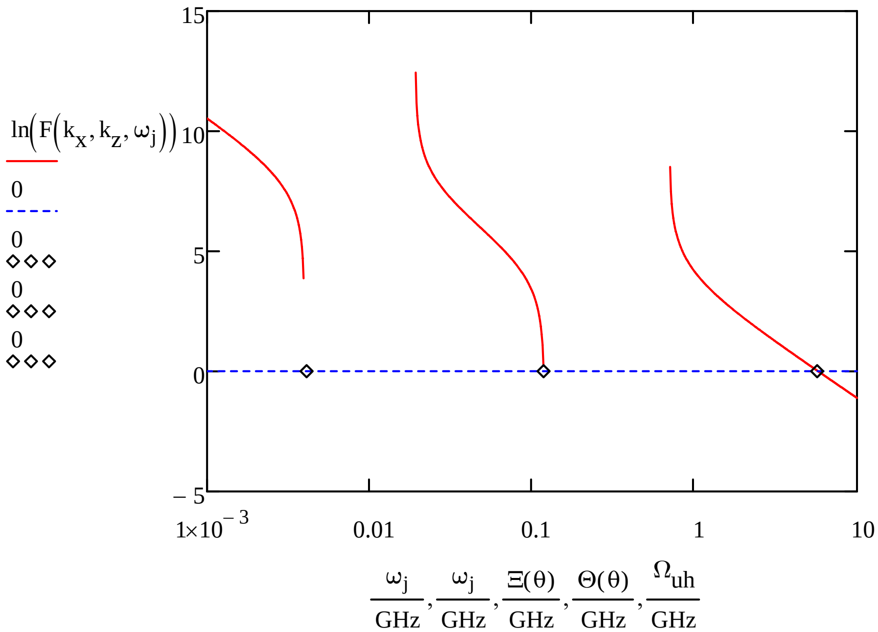
$\Xi(\theta) = 8.324 \times 10^7 \frac{1}{s}$ $\Theta(\theta) = 7.021 \times 10^8 \frac{1}{s}$ $\Omega_{EIC} = 9.318 \times 10^8 \frac{1}{s}$ $\Omega_s(\theta) = 4.323 \times 10^9 \frac{1}{s}$

and see that it is closest to Θ.

Case III. Slightly away from perpendicular (88 degrees)

$\theta := 88 \cdot \text{deg}$ $\quad k_z := k \cdot \cos(\theta)$ $\quad k_x := k \cdot \sin(\theta)$ $\quad k_x = 0.999 \frac{1}{m}$ $\quad k_z = 0.035 \frac{1}{m}$

F(ω) for 88 degree propagation



Mathcad does not plot the log of negative numbers. This curve is missing a sections in the middle. A test shows that these values of F are negative.

Use the root finder to find exactly where F last crosses 1, looking between 10^9 and 10^{10}:

$\text{root}\left(F\left(k_x, k_z, \omega\right) - 1, \omega, 1 \cdot \text{GHz}, 10 \cdot \text{GHz}\right) = 5.759 \times 10^9 \frac{1}{s}$ This is near to ω_{pie} and Ω_{uh}.

$\omega_{pie} = 5.717 \times 10^9 \frac{1}{s}$ $\quad \Omega_{uh} = 5.684 \times 10^9 \frac{1}{s}$

And there is a root between 0.5 and 2 x 10^8

$\text{root}\left(F\left(k_x, k_z, \omega\right) - 1, \omega, 0.5 \times 10^8 \cdot \text{Hz}, 2 \times 10^8 \cdot \text{Hz}\right) = 1.178 \times 10^8 \frac{1}{s}$

which is near to $\Omega_{lh} = 1.153 \times 10^8 \frac{1}{s}$ and nearer to $\Theta(\theta) = 1.178 \times 10^8 \frac{1}{s}$

Last there is a root near 10^6. $A := 1 \cdot \text{MHz}$ $\quad \text{root}\left(F\left(k_x, k_z, A\right) - 1, A\right) = 3.97 \times 10^6 \frac{1}{s}$

which we compare to

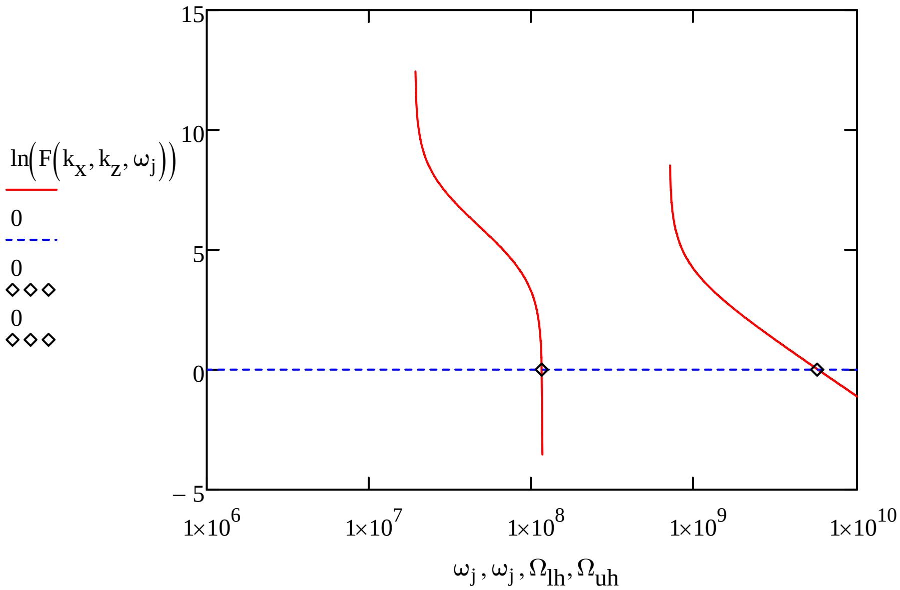
$\Xi(\theta) = 4.108 \times 10^6 \frac{1}{s}$ $\quad \Theta(\theta) = 1.178 \times 10^8 \frac{1}{s}$ $\quad \Omega_s(\theta) = 1.187 \times 10^8 \frac{1}{s}$ $\quad \Omega_{EIC} = 9.318 \times 10^8 \frac{1}{s}$

and see that it is closest to Ξ(θ).

Case IV. Perpendicular propagation (90 degrees)

$\theta := 90\cdot deg$ $k_z := k\cdot\cos(\theta)$ $k_x := k\cdot\sin(\theta)$ $k_x = 1\,\frac{1}{m}$ $k_z = 0\,\frac{1}{m}$

F(ω) for propagation perpendicular to B



Use the root finder to find exactly where F crosses 1, looking between 10^9 and 10^{10}:

$\mathrm{root}\left(F\left(k_x, k_z, \omega\right) - 1, \omega, 1\cdot GHz, 10\cdot GHz\right) = 5.759\cdot GHz$ This is a little larger than $\omega_{pe} = 5.641\times 10^{9}\,\frac{1}{s}$

but it is closer to the upper hybrid frequency: $\Omega_{uh} = 5.684\times 10^{9}\,\frac{1}{s}$

And there is a root near 10^8 which is near

$\mathrm{root}\left(F\left(k_x, k_z, \omega\right) - 1, \omega, 0.5\times 10^{8}\cdot Hz, 2\times 10^{8}\cdot Hz\right) = 1.153\times 10^{8}\,\frac{1}{s}$ $\Omega_{lh} = 1.153\times 10^{8}\,\frac{1}{s}$

For perpendicular propagation, the wave frequencies are the hybrid frequencies.

Try it: Compare results obtained here to the biquadratic formula in Chen's textbook, in problem 4-8a.

Notes:
The frequency Θ(θ) appears in G. Schmidt, *Physics of High Temperature Plasmas* (Academic Press, New York, 1979), Equation 8-56 and exercise 8-8.