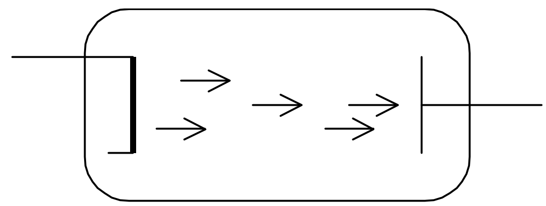
Liouville's equation in 1-d

In the diode vacuum tube electrons are emitted from a cathode and collected at an anode with a positive potential ϕ. This device was used as a rectifier because electrons could only flow one way. No electrons were emitted from the anode.



Electrons are emitted with a Maxwellian distribution at the temperature of the cathode. This Maxwellian is really a half-Maxwellian since electrons are only moving to the right. Liouville's equation helps us answer the question:

What is the distribution function of the electrons after they have been accelerated?
What is the number density?

Liouville 's equation: $$\frac{Df}{Dt} = 0$$

simply says that the convective derivative of the distribution function is zero. In words: if at position x_1 and velocity v_1 the value of the function $f(x_1,v_1,\phi(x_1))$ is C (for example), and the particles move to position x_2 and velocity v_2, then the value of $f(x_2,v_2,\phi(x_2))$ is also C at the new location:

$$f(x_1,v_1,\phi(x_1)) = f(x_2,v_2,\phi(x_2)).$$

Putting in a new value for x_2 is easy, but v_2 must be calculated.
How is this used? We will use the diode as an example in one dimension.

Our beginning point is just outside the cathode surface where $f(x,v,\phi)$ is a half-Maxwellian.

We will need these variables: $m := 1 \quad T := 0.5 \quad q := -1 \quad \phi := 0 \quad n := 1$

where m is the electron mass, T is the cathode temperature, q is the electron charge, ϕ is the electrostatic potential, and n is the number density. The half-Maxwellian in one dimension is:

$$f(v,\phi) := \frac{2}{\sqrt{\pi}} \cdot n\, e^{\frac{-0.5 \cdot m \cdot v^2 - q \cdot \phi}{T}}$$ with the normalization: $$\int_0^\infty f(v,\phi)\, dv = 1$$

The mean velocity of the emitted electrons is

$$v_{mean} := \frac{\int_0^\infty f(v,\phi) \cdot v\, dv}{\int_0^\infty f(v,\phi)\, dv} \qquad v_{mean} = 0.564$$

The flux of electrons leaving the cathode is n v_{mean}.

Note that $\frac{1}{\sqrt{\pi}} = 0.564$ is the first velocity moment of our half-Maxwellian.

The number density as a function of $\phi(x)$:

The electrons are accelerated by the potential drop as they approach the anode.
The number density is found from the velocity integral. BUT: after acceleration through a potential drop ϕ, there is no electron velocity smaller than

$$v_{min}(\phi) := \sqrt{\frac{-2\cdot q\cdot \phi}{m}}$$

so the integral over velocity must start at v_{min}. This is the only "tricky" point in this exercise.
We will use the constancy of $f(x,v,\phi)$, and integrate to get the density

$$n1(\phi) := \frac{2}{\sqrt{\pi}}\cdot \int_{v_{min}(\phi)}^{\infty} e^{\frac{-0.5\cdot m\cdot v^2 - q\cdot\phi}{T}}\,dv$$

At zero potential: $n1(0) = 1$ move to potential $\phi = 1$: $n1(1) = 0.336$

For large potentials, the initial energy is a negligible fraction of the total and the velocity is approximately:

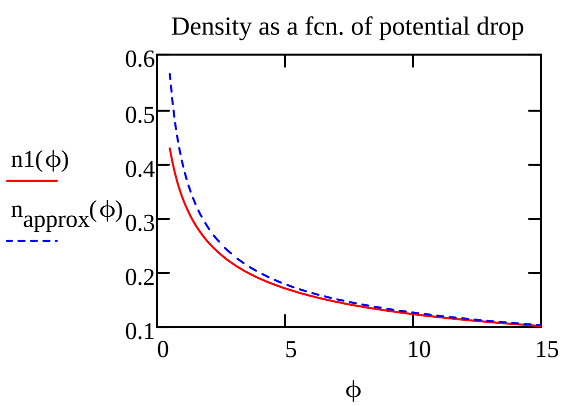
$$v_{approx}(\phi) := \sqrt{\frac{-2\cdot q\cdot \phi}{m}}$$

The continuity equation tells us that the current or flux is a constant.
The density can be found approximately by dividing the flux nv_{mean} by the approximate velocity:

$$n_{approx}(\phi) := \frac{n\cdot v_{mean}}{v_{approx}(\phi)}$$

Let's compare n_{approx} with the accurate result from using Liouville's equation:

Define a range of ϕ $\phi := 0.5, 0.6 .. 15$



This plot compares two ways of finding the density. n1 uses Liouville's equation and is exact. n_{approx} uses the continuity equation, and divides the current by the approximate velocity to obtain the density.

With a change of variables (ϕ becomes a^2) we can find that our integral for n1(ϕ) has a solution in terms of the error function:

$$\frac{2}{\sqrt{\pi}} e^{a^2} \cdot \int_a^{\infty} e^{-x^2} dx \rightarrow -e^{a^2} \cdot (\mathrm{erf}(a) - 1) \text{ simplify } \rightarrow -e^{a^2} \cdot (\mathrm{erf}(a) - 1)$$

However, this is not particularly advantageous because Mathcad can evaluate our integral for n1(ϕ) as easily as it can evaluate the integral for the error function.

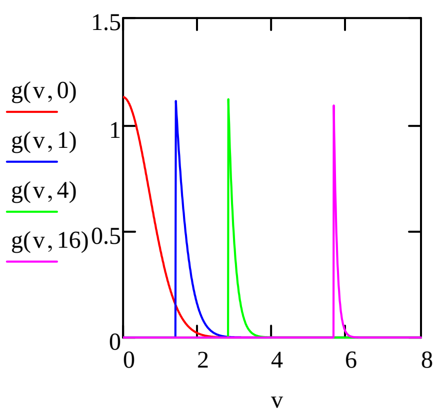
The new distribution function:

$$g(v, \phi) := \mathrm{if}\left(v \geq \sqrt{\frac{-2 \cdot q \cdot \phi}{m}}, \frac{2}{\sqrt{\pi}} \cdot e^{\frac{-0.5 \cdot m \cdot v^2 - q \cdot \phi}{T}}, 0 \right)$$

The *if* function selects the first option if the condition is true, otherwise the second option is used.

The new distribution is zero if the velocity is less than the minimum velocity determined by the potential drop, and it is given by the original distribution function if the velocity is greater.

Define some velocities for a plot: $v := 0, 0.01 .. 8$



As our electron beam accelerates, the minimum velocity determined by the potential drop determines the velocity where the distribution function begins.

Notice that the distribution function is narrower in velocity as the beam is accelerated. The energy spread dE of the beam stays constant. Using E = 0.5 m v^2, we can find that dE = m v dv. This means v dv is a constant and the spread dv varies approximately inversely with the velocity v. Thus a beam appears "colder" in the moving frame after acceleration.

Try it: Make a plot of of g(v,ϕ) as a function of E = 0.5 m v^2. Does the plotted distribution look narrower for larger values of potential? Should it?

Try it: The plot is made for v:=0, 0.01; 8. Change the values of v for the plot to v:=0, 0.05; 8. Why aren't the peaks the same height as before? Hint: change the plot from a line plot to a point plot.